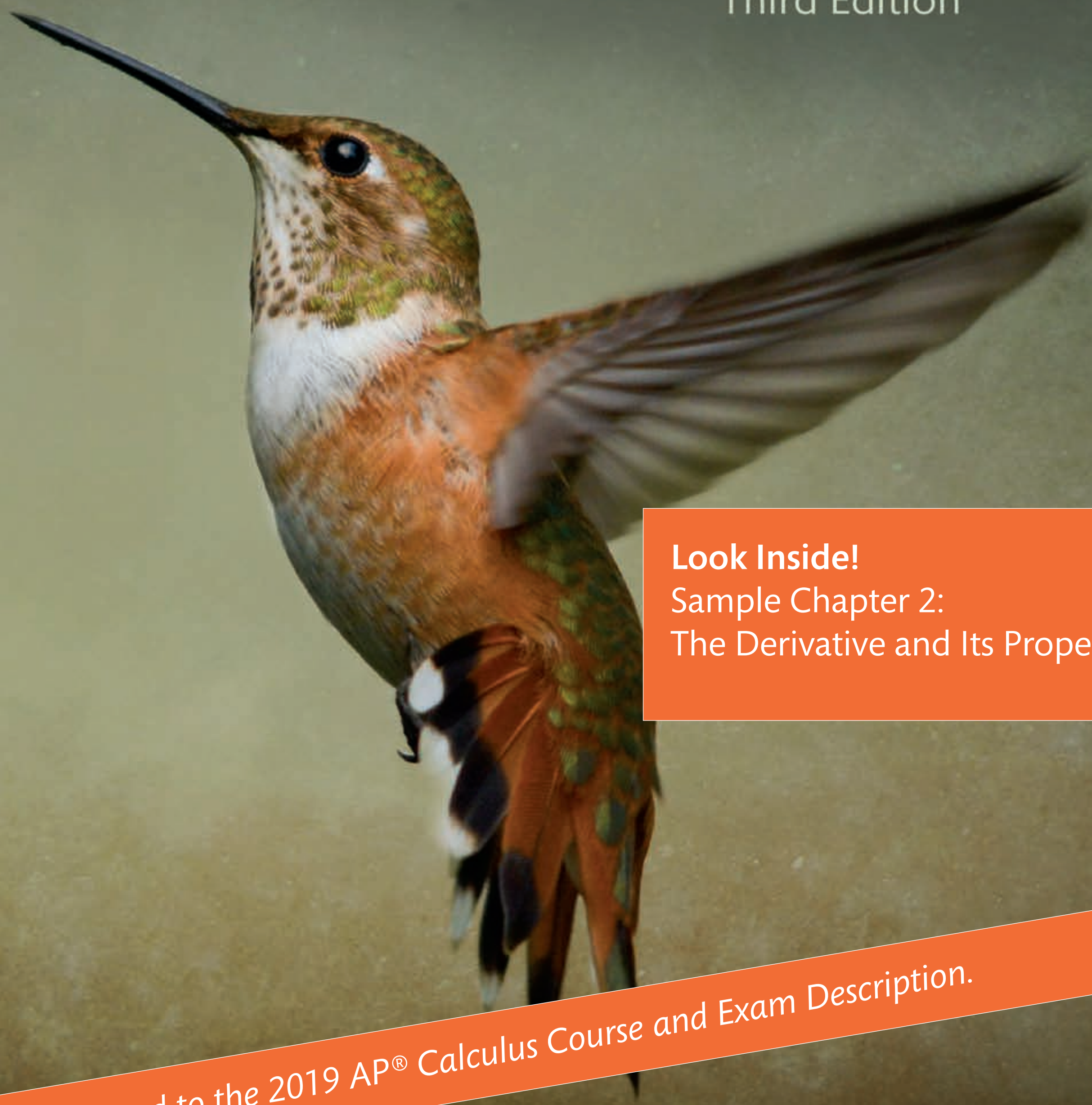
CALCULUS

for the AP® Course

Third Edition



Look Inside!
Sample Chapter 2:
The Derivative and Its Properties

Fully Aligned to the 2019 AP® Calculus Course and Exam Description.

Sullivan | Miranda

Two of the most trusted authors in calculus.

Michael Sullivan

Michael Sullivan, Emeritus Professor of Mathematics at Chicago State University, received a Ph.D. in mathematics from the Illinois Institute of Technology. Before retiring, Mike taught at Chicago State for 35 years, where he honed an approach to teaching and writing that forms the foundation for his textbooks.

Mike has been writing for more than 35 years and currently has 15 books in print. Mike is a member of the American Mathematical Association of Two Year Colleges, the American Mathematical Society, the Mathematical Association of America, and the Textbook and Academic Authors Association. Mike serves on the governing board of TAA and represents TAA on the board of the Authors Coalition of America, a consortium of 22 author/creator organizations in the United States. In 2007, he received the TAA Lifetime Achievement Award.

His influence in the field of mathematics extends to his four children: Kathleen, who teaches college mathematics; Michael III, who also teaches college mathematics, and who is his coauthor on two precalculus series; Dan, who is a sales director for a college textbook publishing company; and Colleen, who teaches middle-school and secondary school mathematics. Twelve grandchildren round out the family. Mike would like to dedicate *Calculus for the AP® Course*, Third Edition, to his four children, 12 grandchildren, and future generations.

Kathleen Miranda

Kathleen Miranda, Ed.D from St. John's University, is an Emeritus Associate Professor of the State University of New York (SUNY), where she taught for 25 years. Kathleen is a recipient of the prestigious New York State Chancellor's Award for Excellence in Teaching, and she particularly enjoys teaching mathematics to underprepared and fearful students. Kathleen has served as director of Curriculum and Assessment Development at SUNY Old Westbury.

In addition to her extensive classroom experience, Kathleen has worked as accuracy reviewer and solutions author on several mathematics textbooks, including Michael Sullivan's *Brief Calculus* and *Finite Mathematics*. Kathleen's goal is to help students unlock the complexities of calculus and appreciate its many applications. Kathleen has four children: Edward, a plastic surgeon in San Francisco; James, an anesthesiologist in Philadelphia; Kathleen, a chemical engineer who directs a biologics division at a major pharmaceutical firm; and Michael, a corporate strategy specialist and entrepreneur. Kathleen would like to dedicate *Calculus for the AP® Course*, Third Edition, to her children and grandchildren.

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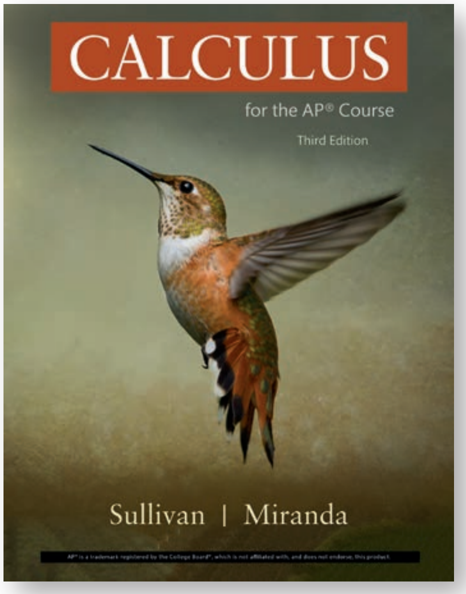
Spread your wings with

The proven AP® Calculus program



Specifically designed to support the needs of AP® students and teachers as well as align with the current AP® Calculus Course and Exam Description (CED), Sullivan and Miranda's *Calculus for the AP® Course*, Third Edition, offers a student-friendly and focused narrative with distinctive features that provide integrated support.

This edition has been carefully developed to adhere to the unit structure and coverage set forth in the College Board's 2019 AP® Calculus CED. Further, it aligns with the CED's overarching structure, meaning every Big Idea, Mathematical Practice, and Student Skill. This edition also aligns with the revised pedagogy of Enduring Understanding, Learning Objective, and Essential Knowledge statements that flow from the three revised Big Ideas.

Written to be read and understood by students as they learn calculus and prepare for either the AP® Calculus AB or BC Exam—the Sullivan and Miranda program offers abundant practice, AP® specific content, distinctive features, and built-in support. The new edition comes complete with BFW's SaplingPlus online-homework platform and a full set of updated teacher resources.

Learn More: **go.bfwpub.com/APCalculus3e**

a powerful new edition.

fully aligned to the 2019 AP® Calculus CED!

AP® Exam Prep is Built within the Text

- AP® Exam Tips appear throughout the text where needed
- *At Section level:* AP® Practice Problems cover content that may appear on the AB or BC versions of the exam
- *At Chapter level:* AP® Review Problems include multiple choice and model FRQs for exam practice all year
- *At Chapter level:* AP® Cumulative Review Problem sets encourage retention of content through cross-chapter review.
- Two AP® Practice Exams
  - 1 full-length AP® Calculus AB practice test, after chapter 8
  - 1 full-length AP® Calculus BC practice test, after chapter 10

COLLEGE BOARD CED	Sullivan and Miranda *Calculus for the AP® course* third edition	Recommendations: College Board Assessments
	Chapter P: Preparing for Calculus	
Unit 1: Limits and Continuity	**Unit 1: Limits and Continuity** Chapter 1: Limits and Continuity	*Assign Personal Progress Check #1*
Unit 2: Differentiation: Definition and Fundamental Properties	**Unit 2: Differentiation: Definition and Fundamental Properties** Chapter 2: The Derivative and Its Properties	*Assign Personal Progress Check #2*
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	**Unit 3: Differentiation: Composite, Implicit, and Inverse Functions** Chapter 3: The Derivative of Composite, Implicit, and Inverse Functions	*Assign Personal Progress Check #3*
Unit 4: Contextual Applications of Differentiation	**Unit 4: Contextual Applications of Differentiation** Chapter 4: Applications of the Derivative Part 1	*Assign Personal Progress Check #4*
Unit 5: Analytical Applications of Differentiation	**Unit 5: Analytical Applications of Differentiation** Chapter 5: Applications of the Derivative, Part 2	*Assign Personal Progress Check #5*
Unit 6: Integration and Accumulation of Change	**Unit 6: Integration and Accumulation of Change** Chapter 6: PART 1 The Integral Chapter 6: PART 2 Techniques of Integration	*Assign Personal Progress Check #6*
Unit 7: Differential Equations	**Unit 7: Differential Equations** Chapter 7: Differential Equations	*Assign Personal Progress Check #7*
Unit 8: Applications of Integration	**Unit 8: Applications of Integration** Chapter 8: Applications of the Integral	*Assign Personal Progress Check #8*
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions (BC ONLY)	**Unit 9: Parametric Equations; Polar Coordinates; and Vector-Valued Function** Chapter 9: Parametric Equations; Polar Coordinates; Vector Functions	*Assign Personal Progress Check #9*
Unit 10: Infinite Series and Sequences (BC ONLY)	**Unit 10: Infinite Series and Sequences** Chapter 10: Infinite Series	*Assign Personal Progress Check #10*

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Support Where Your Students Need It Most.

RECALL gives a quick refresher of key results used in theorems, definitions, and examples.

NEED TO REVIEW? shows you where in the book to find concepts that you want to refresh. Each of these is on-the-spot help to send you to the right information.

AP® EXAM TIPS offer advice and tips on how to succeed on the exam.

ORIGINS point out the life story and key discoveries of people who helped in the development of calculus.

CAUTION notes warn you about potential misconceptions and pitfalls.

EXAMPLE 2 Finding an Equation of a Tangent Line

The function $f(x)=x^3+2x-4$ is one-to-one and has an inverse function g. Find an equation of the tangent line to the graph of g at the point $(8, 2)$ on g.

Solution

The slope of the tangent line to the graph of g at the point $(8, 2)$ is $g'(8)$. Since f and g are inverses, $g(8)=2$ and $f(2)=8$. Now use (1) to find $g'(8)$.

$$g'(8)=\frac{1}{f'(2)} \qquad g'(y_0)=\frac{1}{f'(x_0)};\ x_0=2;\ y_0=8$$

$$=\frac{1}{14} \qquad f'(x)=3x^2+2;\ f'(2)=3\cdot 2^2+2=14$$

Now use the point-slope form of an equation of a line to find an equation of the tangent line to g at $(8, 2)$.

$$y-2=\frac{1}{14}(x-8)$$
$$14y-28=x-8$$
$$14y-x=20$$

The line $14y-x=20$ is tangent to the graph of g at the point $(8, 2)$. ■

NOW WORK Problem 51 and AP® Practice Problem 7.

RECALL If the functions f and g are inverses and if $f(a)=b$, then $g(b)$ exists, and $g(b)=a$.

2 Find the Derivative of the Inverse Trigonometric Functions

Table 1 lists the inverse trigonometric functions and their domains.

NEED TO REVIEW? Inverse trigonometric functions are discussed in Section P.7, pp. 60–64.

AP® EXAM TIP

The exam often uses $\arcsin x$ instead of $\sin^{-1} x$, $\arccos x$ instead of $\cos^{-1} x$, and so on. Be familiar with both notations.

TABLE 1 The Domain of the Inverse Trigonometric Functions

f	Restricted Domain	f^{-1}	Domain
$f(x)=\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$f^{-1}(x)=\sin^{-1} x$	$[-1, 1]$
$f(x)=\cos x$	$[0, \pi]$	$f^{-1}(x)=\cos^{-1} x$	$[-1, 1]$
$f(x)=\tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		
$f(x)=\csc x$	$\left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$		
$f(x)=\sec x$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$		
$f(x)=\cot x$	$(0, \pi)$		

CAUTION Keep in mind that $f^{-1}(x)$ represents the inverse function of f and not the reciprocal function. That is, $f^{-1}(x)\neq\frac{1}{f(x)}$.

To find the derivative of $y=\sin^{-1} x$, $-1\le x\le 1$, $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, we write $\sin y=x$ and differentiate implicitly with respect to x.

$$\frac{d}{dx}\sin y=\frac{d}{dx}x$$

$$\cos y\cdot\frac{dy}{dx}=1 \qquad \frac{d}{dx}\sin y=\cos y\frac{dy}{dx} \text{ using the Chain Rule.}$$

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

NOTE Alternatively, we can use the Derivative of an Inverse Function. If $y=\sin^{-1} x$, then $x=\sin y$, and

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\cos y}$$

provided $\cos y\neq 0$.

Giuseppe Nogari/Getty Images

ORIGINS Archimedes (287–212 BC) was a Greek astronomer, physicist, engineer, and mathematician. His contributions include the Archimedes screw, which is still used to lift water or grain, and the Archimedes principle, which states that the weight of an object can be determined by measuring the volume of water it displaces. Archimedes used the principle to measure the amount of gold in the king's crown; modern-day cooks can use it to measure the amount of butter to add to a recipe. Archimedes is also credited with using the Method of Exhaustion to find the area under a parabola and to approximate the value of π.

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Exercises. Practice Makes Perfect.

Start with the **Concepts and Vocabulary** to check your comprehension of the main points.

Want to master calculus concepts? **Assess Your Understanding** exercises appear at the end of each section. Selected answers appear in the back of the book.

2.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\frac{d}{dx}\cos x = \sin x$
2. *True or False* $\frac{d}{dx}\tan x = \cot x$
3. *True or False* $\frac{d^2}{dx^2}\sin x = -\sin x$
4. *True or False* $\frac{d}{dx}\sin\frac{\pi}{3} = \cos\frac{\pi}{3}$

Skill Building problems help you to develop computation skills and the ability to select the best approach to solve a problem.

Skill Building

In Problems 5–38, find y'.

PAGE 208 5. $y = x - \sin x$ 6. $y = \cos x - x^2$

7. $y = \tan x + \cos x$ 8. $y = \sin x - \tan x$

9. $y = 3\sin\theta - 2\cos\theta$ 10. $y = 4\tan\theta + \sin\theta$

11. $y = \sin x\cos x$ 12. $y = \cot x\tan x$

PAGE 209 13. $y = t\cos t$ 14. $y = t^2\tan t$

PAGE 210 15. $y = e^x\tan x$ 16. $y = e^x\sec x$

17. $y = \pi\sec u\tan u$ 18. $y = \pi u\tan u$

19. $y = \frac{\cot x}{x}$ 20. $y = \frac{\csc x}{x}$

21. $y = x^2\sin x$ 22. $y = t^2\tan t$

23. $y = t\tan t - \sqrt{3}\sec t$ 24. $y = x\sec x + \sqrt{2}\cot x$

25. $y = \frac{\sin\theta}{1-\cos\theta}$ 26. $y = \frac{x}{\cos x}$

27. $y = \frac{\sin t}{1+t}$ 28. $y = \frac{\tan u}{1+u}$

PAGE 208 29. $y = \frac{\sin x}{e^x}$ 30. $y = \frac{\cos x}{e^x}$

In Problems 39–50, find y''.

39. $y = \sin x$ 40. $y = \cos x$

41. $y = \tan\theta$ 42. $y = \cot\theta$

43. $y = t\sin t$ 44. $y = t\cos t$

PAGE 210 45. $y = e^x\sin x$ 46. $y = e^x\cos x$

47. $y = 2\sin u - 3\cos u$ 48. $y = 3\sin u + 4\cos u$

49. $y = a\sin x + b\cos x$ 50. $y = a\sec\theta + b\tan\theta$

In Problems 51–56:

(a) Find an equation of the tangent line to the graph of f at the indicated point.

(b) Graph the function and the tangent line.

51. $f(x) = \sin x$ at $(0, 0)$ 52. $f(x) = \cos x$ at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$

53. $f(x) = \tan x$ at $(0, 0)$ 54. $f(x) = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$

55. $f(x) = \sin x + \cos x$ at $\left(\frac{\pi}{4}, \sqrt{2}\right)$

56. $f(x) = \sin x - \cos x$ at $\left(\frac{\pi}{4}, 0\right)$

In Problems 57–60:

(a) Find all points on the graph of f where the tangent line is horizontal.

(b) Graph the function and the horizontal tangent lines on the interval $[-2\pi, 2\pi]$.

PAGE 209 57. $f(x) = 2\sin x + \cos x$ 58. $f(x) = \cos x - \sin x$

59. $f(x) = \sec x$ 60. $f(x) = \csc x$

Applications and Extensions

In Problems 61 and 62, find the nth derivative of each function.

61. $f(x) = \sin x$ 62. $f(\theta) = \cos\theta$

63. What is $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\frac{\pi}{2}}{h}$?

64. What is $\lim_{h\to 0}\frac{\sin(\pi+h) - \sin\pi}{h}$?

Icons identify problems that require a **graphing calculator**.

Once you've mastered computational skills, tackle the **Applications and Extensions** problems that are applied or extend the concepts of the section.

Exam prep all year. Each section that covers content that may appear on the exam has **AP® Practice Problems**. Practicing all year will pay off when you are more confident taking the exam.

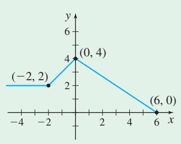
Preparing for the AP® Exam

AP® Practice Problems

PAGE 176 1. The function $f(x) = \begin{cases} x^2 - ax & \text{if } x \le 1 \\ ax + b & \text{if } x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

(A) -3 (B) -2 (C) 0 (D) 2

PAGE 172 2. The graph of the function f, given below, consists of three line segments. Find $\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$.



(A) -1 (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) does not exist

PAGE 179 3. If $f(x) = \begin{cases} \frac{x^2-25}{x-5} & \text{if } x \ne 5 \\ 5 & \text{if } x = 5 \end{cases}$ which of the following statements about f are true?

I. $\lim_{x\to 5} f$ exists.
II. f is continuous at $x = 5$.
III. f is differentiable at $x = 5$.

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 179 4. Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

I. f has at least 2 zeros.
II. f is continuous on the closed interval $[-1, 7]$.
III. For some c, $0 < c < 7$, $f(c) = -2$.

(A) I only (B) I and II only
(C) II and III only (D) I, II, and III

PAGE 176 5. If $f(x) = |x|$, which of the following statements about f are true?

I. f is continuous at 0.
II. f is differentiable at 0.
III. $f(0) = 0$.

(A) I only (B) III only
(C) I and III only (D) I, II, and III

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The Style Your Students Love.

Sullivan and Miranda's Calculus for the AP® Course, third edition was written at an accessible level so that students can handle the concepts and the worked examples.

- The textbook offers focused descriptions with integrated support written to be read and understood by students as they learn calculus and prepare for the AP® Exam.
- Sullivan and Miranda's distinctive worked examples with their NOW WORK prompt enhance good habits by directing students to a problem in the section exercise set that mirrors the worked example and provides immediate practice and reinforcement. This proven backward/forward navigation system efficiently directs students from exercise to example and vice versa.
- The many examples have worked solutions to serve as models for your students. Each is carefully calibrated in difficulty throughout the section to help students attain higher levels of skill in dealing with complexity.
- Selected worked examples are supported by short video CalcClips that provide a step-by-step working out of an example that mirrors the in-text example.
- All AP® Review Problems are accompanied by short video clips that provide step-by-step support through the AP® problem.
- All AP® Cumulative Review Problems are accompanied by short video clips that provide step-by-step support through the AP® problem.

Extra support when you need it. Examples marked with the Calc Clip button CALC CLIP are supported by short video clips that walk you through each step in the process of solving a similar problem.

CALC CLIP **EXAMPLE 2** Using Implicit Differentiation to Find an Equation of a Tangent Line

Find an equation of the tangent line to the graph of the ellipse $3x^2+4y^2=2x$ at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$. Graph the equation and the tangent line on the same set of coordinate axes.

Solution

First use implicit differentiation to find the slope of the tangent line.

Assuming y is a differentiable function of x, differentiate both sides with respect to x.

$$\frac{d}{dx}(3x^2+4y^2)=\frac{d}{dx}(2x) \qquad \text{Differentiate both sides with respect to } x.$$

$$\frac{d}{dx}(3x^2)+\frac{d}{dx}(4y^2)=2 \qquad \text{Use the Sum Rule.}$$

$$3\frac{d}{dx}x^2+4\frac{d}{dx}y^2=2 \qquad \text{Use the Constant Multiple Rule.}$$

$$6x+4\cdot 2y\frac{dy}{dx}=2 \qquad \frac{d}{dx}y^2=2y\frac{dy}{dx}$$

$$6x+8y\frac{dy}{dx}=2 \qquad \text{Simplify.}$$

$$\frac{dy}{dx}=\frac{2-6x}{8y}=\frac{1-3x}{4y} \qquad \text{Solve for } \frac{dy}{dx}.$$

provided $y \neq 0$.

Now evaluate $\frac{dy}{dx}=\frac{1-3x}{4y}$ at $\left(\frac{1}{2}, -\frac{1}{4}\right)$.

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},-\frac{1}{4}\right)}=\left.\frac{1-3x}{4y}\right|_{\left(\frac{1}{2},-\frac{1}{4}\right)}=\frac{1-3\cdot\frac{1}{2}}{4\cdot\left(-\frac{1}{4}\right)}=\frac{1}{2}$$

The slope of the tangent line to the graph of $3x^2+4y^2=2x$ at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$ is $\frac{1}{2}$. An equation of the tangent line is

$$y+\frac{1}{4}=\frac{1}{2}\left(x-\frac{1}{2}\right)$$

$$y=\frac{1}{2}x-\frac{1}{2}$$

Figure 2 shows the graph of the ellipse $3x^2+4y^2=2x$ and the graph of the tangent line $y=\frac{1}{2}x-\frac{1}{2}$ at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$. ■

NOW WORK Problems 15 and 53 and AP® Practice Problems 3 and 5.

NOTE The notation $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$ is used when the derivative $\frac{dy}{dx}$ is to be evaluated at a point (x_0, y_0).

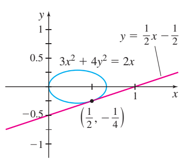

Figure 2

Information-rich examples. Worked examples provide step-by-step instruction. Look for annotations in blue that show you what formula or reasoning is involved in solving the problem.

Access to practice. After working through an example, try the NOW WORK Practice Problems and **AP® Practice Problems** at the end of the section to master the concepts.

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Sullivan and Miranda's Calculus for the AP® Course, Third Edition, comes complete with our robust online-homework and digital platform, **SaplingPlus**. Every problem includes targeted feedback that tailors itself specifically to each student's answer or response. This essential and real-time guidance makes every problem count, allowing students to learn even when they get a solution incorrect. SaplingPlus features:

- Over 2000+ questions, many of them **algorithmically driven**
- Many review questions from the textbook, in **prebuilt assignments**
- **Targeted feedback**, providing students with real-time guidance on every problem
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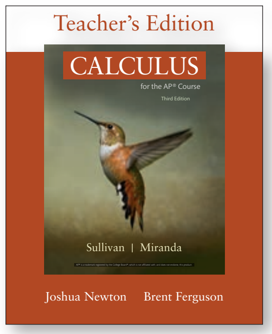
A Complete Teacher Resource Package.



The *Sullivan and Miranda's Calculus for the AP® Course* Third Edition program will include a **completely revamped** teacher resources suite that is fully aligned to the 2019 CED.

- **The NEW Teacher's Edition** includes dozens of features built for the teacher to master their instruction. This includes an in-depth correlation in the "Blue Pages" before each chapter, **aligning each chapter to the 2019 AP® Calculus CED**—as well as realigned pacing guides chapter by chapter for the AB and BC courses. Our authors have done the hard work of ensuring close correlation so that the teacher can focus on teaching.
- **The Teacher's Resource Flashdrive** includes all of the unmatched and abundant resources you've come to expect with BFW's AP® programs: videos, worksheets, solution manuals, activities, flashcards, prepared exams, and more—**New to the third edition: *Desmos* Activities!**
- **ExamView® Assessment Suite**: This hallmark resource includes over 2000 additional questions, written by a team of AP® and content experts.

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Chapter 2

The Derivative and Its Properties

Unit 2

Differentiation: Definition and Fundamental Properties

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The *Apollo* Lunar Module

"One Giant Leap for Mankind"

On May 25, 1961, in a special address to Congress, U.S. President John F. Kennedy proposed the goal "before this decade is out, of landing a man on the Moon and returning him safely to the Earth." Roughly eight years later, on July 16, 1969, a Saturn V rocket launched from the Kennedy Space Center in Florida, carrying the *Apollo 11* spacecraft and three astronauts—Neil Armstrong, Buzz Aldrin, and Michael Collins—bound for the Moon.

The *Apollo* spacecraft had three parts: the Command Module with a cabin for the three astronauts; the Service Module that supported the Command Module with propulsion, electrical power, oxygen, and water; and the Lunar Module for landing on the Moon. After its launch, the spacecraft traveled for three days until it entered into lunar orbit. Armstrong and Aldrin then moved into the Lunar Module, which they landed in the flat expanse of the Sea of Tranquility. After more than 21 hours, the first humans to touch the surface of the Moon crawled into the Lunar Module and lifted off to rejoin the Command Module, which Collins had been piloting in lunar orbit. The three astronauts then headed back to Earth, where they splashed down in the Pacific Ocean on July 24.

Explore some of the physics at work that allowed engineers and pilots to successfully maneuver the Lunar Module to the Moon's surface in the Chapter 2 Project on page 215.

Chapter 2 opens by returning to the tangent problem to find an equation of the tangent line to the graph of a function f at a point $P=(c, f(c))$. Remember in Section 1.1 we found that the slope of a tangent line was a limit,

$$m_{\tan} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

This limit turns out to be one of the most significant ideas in calculus, the *derivative*.

In this chapter, we introduce interpretations of the derivative, treat the derivative as a function, and consider some properties of the derivative. By the end of the chapter, you will have a collection of basic derivative formulas and derivative rules that will be used throughout your study of calculus.

AP® EXAM TIP

The derivative is an important concept in the AP® Calculus curriculum.

2.1 Rates of Change and the Derivative

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Find equations for the tangent line and the normal line to the graph of a function (p. 162)**
2. **Find the rate of change of a function (p. 163)**
3. **Find average velocity and instantaneous velocity (p. 164)**
4. **Find the derivative of a function at a number (p. 166)**

In Chapter 1, we discussed the tangent problem: *Given a function f and a point P on its graph, what is the slope of the tangent line to the graph of f at P*? See Figure 1, where ℓ_T is the tangent line to the graph of f at the point $P=(c, f(c))$.

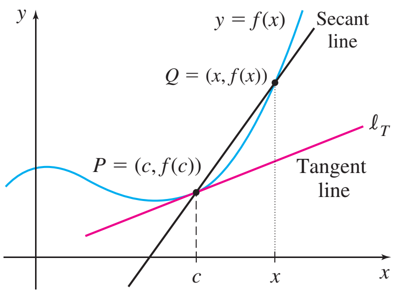


Figure 1 $m_{\sec}$ = slope of the secant line.

The tangent line ℓ_T to the graph of f at P must contain the point P. Since finding the slope requires two points, and we have only one point on the tangent line ℓ_T, we reason as follows.

Suppose we choose any point $Q=(x, f(x))$, other than P, on the graph of f. (Q can be to the left or to the right of P; we chose Q to be to the right of P.) The line containing the points $P=(c, f(c))$ and $Q=(x, f(x))$ is a secant line of the graph of f. The slope $m_{\sec}$ of this secant line is

$$\boxed{m_{\sec} = \frac{f(x)-f(c)}{x-c}} \qquad (1)$$

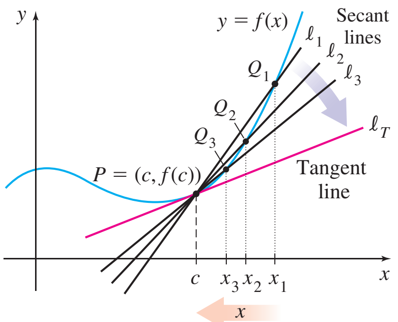


Figure 2 $m_{\tan} = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$

Figure 2 shows three different points Q_1, Q_2, and Q_3 on the graph of f that are successively closer to the point P, and three associated secant lines ℓ_1, ℓ_2, and ℓ_3. The closer the points Q are to the point P, the closer the secant lines are to the tangent line ℓ_T. The line ℓ_T, the *limiting position* of these secant lines, is the *tangent line to the graph* of f at P.

If the limiting position of the secant lines is the tangent line, then the limit of the slopes of the secant lines should equal the slope of the tangent line. Notice in Figure 2 that as the points Q_1, Q_2, and Q_3 move closer to the point P, the numbers x get closer to c. So, equation (1) suggests that

$$\begin{aligned} m_{\tan} &= \text{Slope of the tangent line to } f \text{ at } P \\ &= \text{Limit of } \frac{f(x)-f(c)}{x-c} \text{ as } x \text{ gets closer to } c \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \end{aligned}$$

provided the limit exists.

DEFINITION Tangent Line

The **tangent line** to the graph of f at a point P is the line containing the point $P=(c, f(c))$ and having the slope

$$\boxed{m_{\tan}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}} \qquad (2)$$

provided the limit exists.

NOTE It is possible that the limit in (2) does not exist. The geometric significance of this is discussed in the next section.

The limit in equation (2) that defines the slope of the tangent line occurs so frequently that it is given a special notation $f'(c)$, read, "f prime of c," and called **prime notation**:

$$\boxed{f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}} \qquad (3)$$

1 Find Equations for the Tangent Line and the Normal Line to the Graph of a Function

THEOREM Equation of a Tangent Line

If $m_{\tan}=f'(c)$ exists, then an equation of the tangent line to the graph of a function $y=f(x)$ at the point $P=(c, f(c))$ is

$$\boxed{y-f(c)=f'(c)(x-c)}$$

RECALL Two lines, neither of which is horizontal, with slopes m_1 and m_2, respectively, are perpendicular if and only if

$$m_1=-\frac{1}{m_2}$$

The line perpendicular to the tangent line at a point P on the graph of a function f is called the **normal line** to the graph of f at P.

An equation of the normal line to the graph of a function $y=f(x)$ at the point $P=(c, f(c))$ is

$$\boxed{y-f(c)=-\frac{1}{f'(c)}(x-c)}$$

provided $f'(c)$ exists and is not equal to zero. If $f'(c)=0$, the tangent line is horizontal, the normal line is vertical, and the equation of the normal line is $x=c$.

AP® EXAM TIP

Problems on the exam often ask about the tangent line and the normal line.

CALC CLIP

EXAMPLE 1 Finding Equations for the Tangent Line and the Normal Line

(a) Find the slope of the tangent line to the graph of $f(x)=x^2$ at the point $(-2, 4)$.

(b) Use the result from (a) to find an equation of the tangent line at the point $(-2, 4)$.

(c) Find an equation of the normal line to the graph of f at the point $(-2, 4)$.

(d) Graph f, the tangent line to f at $(-2, 4)$, and the normal line to f at $(-2, 4)$ on the same set of axes.

RECALL One way to find the limit of a quotient when the limit of the denominator is 0 is to factor the numerator and divide out common factors.

Solution

(a) At the point $(-2, 4)$, the slope of the tangent line is

$$f'(-2)=\lim_{x\to -2}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to -2}\frac{x^2-(-2)^2}{x+2}=\lim_{x\to -2}\frac{x^2-4}{x+2}$$

$$=\lim_{x\to -2}(x-2)=-4$$

NEED TO REVIEW? The point-slope form of a line is discussed in Appendix A.3, p. A19.

(b) We use the result from (a) and the point-slope form of an equation of a line to obtain an equation of the tangent line. An equation of the tangent line containing the point $(-2, 4)$ is

$$y-4=f'(-2)[x-(-2)] \qquad \text{Point-slope form of an equation of the tangent line.}$$
$$y-4=-4\cdot(x+2) \qquad f(-2)=4;\quad f'(-2)=-4$$
$$y=-4x-4 \qquad \text{Simplify.}$$

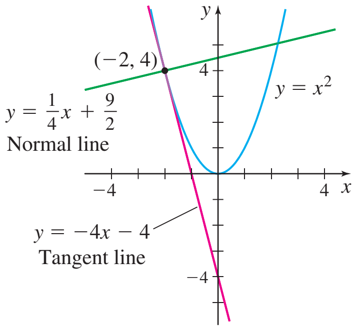


Figure 3 $f(x)=x^2$

(c) Since the slope of the tangent line to f at $(-2, 4)$ is -4, the slope of the normal line to f at $(-2, 4)$ is $\frac{1}{4}$.

Using the point-slope form of an equation of a line, an equation of the normal line is

$$y-4=\frac{1}{4}(x+2)$$
$$y=\frac{1}{4}x+\frac{9}{2}$$

(d) The graphs of f, the tangent line to the graph of f at the point $(-2, 4)$, and the normal line to the graph of f at $(-2, 4)$ are shown in Figure 3. ■

NOW WORK Problem 11 and AP® Practice Problems 1 and 5.

2 Find the Rate of Change of a Function

Everything in nature changes. Examples include climate change, change in the phases of the Moon, and change in populations. To describe natural processes mathematically, the ideas of change and rate of change are often used.

Recall that the average rate of change of a function $y=f(x)$ from c to x is given by

$$\boxed{\text{Average rate of change}=\frac{f(x)-f(c)}{x-c} \qquad x\neq c}$$

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f at c is the limit as x approaches c of the average rate of change. Symbolically, the instantaneous rate of change of f at c is

$$\boxed{\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}$$

provided the limit exists.

IN WORDS
- An average rate of change describes behavior over an interval.
- An instantaneous rate of change describes behavior at a number.

The expression "instantaneous rate of change" is often shortened to *rate of change*. Using prime notation, the **rate of change** of f at c is $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$.

EXAMPLE 2 Finding a Rate of Change

Find the rate of change of the function $f(x)=x^2-5x$ at:

(a) $c=2$

(b) Any real number c

Solution

(a) For $c=2$,

$$f(x)=x^2-5x \qquad \text{and} \qquad f(2)=2^2-5\cdot2=-6$$

The rate of change of f at $c = 2$ is

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{(x^2-5x)-(-6)}{x-2} = \lim_{x\to 2}\frac{x^2-5x+6}{x-2}$$
$$= \lim_{x\to 2}\frac{(x-2)(x-3)}{x-2} = \lim_{x\to 2}(x-3) = -1$$

(b) If c is any real number, then $f(c) = c^2 - 5c$, and the rate of change of f at c is

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{(x^2-5x)-(c^2-5c)}{x-c} = \lim_{x\to c}\frac{(x^2-c^2)-5(x-c)}{x-c}$$
$$= \lim_{x\to c}\frac{(x-c)(x+c)-5(x-c)}{x-c} = \lim_{x\to c}\frac{(x-c)(x+c-5)}{x-c}$$
$$= \lim_{x\to c}(x+c-5) = 2c-5$$
■

NOW WORK Problem 17 and AP® Practice Problem 3.

EXAMPLE 3 Finding the Rate of Change in a Biology Experiment

In a metabolic experiment, the mass M of glucose decreases according to the function

$$M(t) = 4.5 - 0.03t^2$$

where M is measured in grams (g) and t is the time in hours (h). Find the reaction rate $M'(t)$ at $t = 1$ h.

Solution

The reaction rate at $t = 1$ is $M'(1)$.

$$M'(1) = \lim_{t\to 1}\frac{M(t)-M(1)}{t-1} = \lim_{t\to 1}\frac{(4.5-0.03t^2)-(4.5-0.03)}{t-1}$$
$$= \lim_{t\to 1}\frac{-0.03t^2+0.03}{t-1} = \lim_{t\to 1}\frac{(-0.03)(t^2-1)}{t-1} = \lim_{t\to 1}\frac{(-0.03)(t-1)(t+1)}{t-1}$$
$$= -0.03\cdot 2 = -0.06$$

The reaction rate at $t = 1$ h is -0.06 g/h. That is, the mass M of glucose at $t = 1$ h is decreasing at the rate of 0.06 g/h. ■

NOW WORK Problem 43.

3 Find Average Velocity and Instantaneous Velocity

Average velocity is a physical example of an average rate of change. For example, consider an object moving along a horizontal line with the positive direction to the right, or moving along a vertical line with the positive direction upward. Motion along a line is referred to as **rectilinear motion**. The object's location at time $t = 0$ is called its **initial position**. The initial position is usually marked as the origin O on the line. See Figure 4. We assume the position s at time t of the object from the origin is given by a function $s = f(t)$. Here s is the signed, or directed, distance (using some measure of distance such as centimeters, meters, feet, etc.) of the object from O at time t (in seconds or hours). The function f is usually called the **position function** of the object.

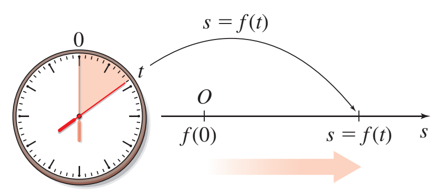


Figure 4 t is the travel time. s is the signed distance of the object from the origin at time t.

DEFINITION Average Velocity

The signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$. If at time t_0 the object is at $s_0 = f(t_0)$ and at time t_1 the object is at $s_1 = f(t_1)$, then the change in time is $\Delta t = t_1 - t_0$ and the change in position is $\Delta s = s_1 - s_0 = f(t_1) - f(t_0)$. The average rate of change of position with respect to time is

$$\boxed{\frac{\Delta s}{\Delta t} = \frac{f(t_1)-f(t_0)}{t_1-t_0} \qquad t_1 \neq t_0}$$

and is called the **average velocity** of the object over the interval $[t_0, t_1]$. See Figure 5.

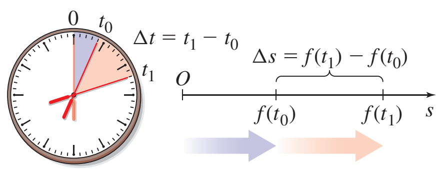


Figure 5 The average velocity is $\dfrac{\Delta s}{\Delta t}$.

EXAMPLE 4 Finding Average Velocity

Scott Prokop/Shutterstock

The Mike O'Callaghan–Pat Tillman Memorial Bridge spanning the Colorado River opened on October 16, 2010. Having a span of 1900 ft, it is the longest arch bridge in the Western Hemisphere, and its roadway is 890 ft above the Colorado River.

If a rock falls from the roadway, the function $s = f(t) = 16t^2$ gives the distance s, in feet, that the rock falls after t seconds for $0 \le t \le 7.458$. Here 7.458 s is the approximate time it takes the rock to fall 890 ft into the river. The average velocity of the rock during its fall is

$$\frac{\Delta s}{\Delta t} = \frac{f(7.458) - f(0)}{7.458 - 0} = \frac{890 - 0}{7.458} \approx 119.335 \text{ ft/s}$$ ■

NOW WORK AP® Practice Problem 7.

NOTE Here the motion occurs along a vertical line with the positive direction downward.

The average velocity of the rock in Example 4 approximates the average velocity over the interval [0, 7.458]. But the average velocity does not tell us about the velocity at any particular instant of time. That is, it gives no information about the rock's *instantaneous velocity*.

We can investigate the instantaneous velocity of the rock, say, at $t = 3$ s, by computing average velocities for short intervals of time beginning at $t = 3$. First we compute the average velocity for the interval beginning at $t = 3$ and ending at $t = 3.5$. The corresponding distances the rock has fallen are

$$f(3) = 16 \cdot 3^2 = 144 \text{ ft} \qquad \text{and} \qquad f(3.5) = 16 \cdot 3.5^2 = 196 \text{ ft}$$

Then $\Delta t = 3.5 - 3.0 = 0.5$, and during this 0.5-s interval,

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{f(3.5) - f(3)}{3.5 - 3} = \frac{196 - 144}{0.5} = 104 \text{ ft/s}$$

Table 1 shows average velocities of the rock for smaller intervals of time.

TABLE 1

Time interval	Start $t_0 = 3$	End t	Δt	$\frac{\Delta s}{\Delta t} = \frac{f(t) - f(t_0)}{t - t_0} = \frac{16t^2 - 144}{t - 3}$
[3, 3.1]	3	3.1	0.1	$\frac{\Delta s}{\Delta t} = \frac{f(3.1) - f(3)}{3.1 - 3} = \frac{16 \cdot 3.1^2 - 144}{0.1} = 97.6$
[3, 3.01]	3	3.01	0.01	$\frac{\Delta s}{\Delta t} = \frac{f(3.01) - f(3)}{3.01 - 3} = \frac{16 \cdot 3.01^2 - 144}{0.01} = 96.16$
[3, 3.0001]	3	3.0001	0.0001	$\frac{\Delta s}{\Delta t} = \frac{f(3.0001) - f(3)}{3.0001 - 3} = \frac{16 \cdot 3.0001^2 - 144}{0.0001} = 96.0016$

The average velocity of 96.0016 over the time interval $\Delta t = 0.0001$ s should be very close to the instantaneous velocity of the rock at $t = 3$ s. As Δt gets closer to 0, the average velocity gets closer to the instantaneous velocity. So, to obtain the instantaneous velocity at $t = 3$ precisely, we use the limit of the average velocity as Δt approaches 0 or, equivalently, as t approaches 3.

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 3} \frac{f(t) - f(3)}{t - 3} = \lim_{t \to 3} \frac{16t^2 - 16 \cdot 3^2}{t - 3} = \lim_{t \to 3} \frac{16(t^2 - 9)}{t - 3}$$

$$= \lim_{t \to 3} \frac{16(t - 3)(t + 3)}{t - 3} = \lim_{t \to 3} [16(t + 3)] = 96$$

The rock's instantaneous velocity at $t = 3$ s is 96 ft/s.

We generalize this result to obtain a definition for instantaneous velocity.

DEFINITION Instantaneous Velocity

If $s = f(t)$ is the position function of an object at time t, the **instantaneous velocity** v of the object at time t_0 is defined as the limit of the average velocity $\frac{\Delta s}{\Delta t}$ as Δt approaches 0. That is,

$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}}$$

provided the limit exists.

IN WORDS

- Average velocity is measured over an interval of time.
- Instantaneous velocity is measured at a particular instant of time.

We usually shorten "instantaneous velocity" and just use the word "velocity."

NOW WORK Problem 31.

EXAMPLE 5 Finding Velocity

Find the velocity v of the falling rock from Example 4 at:

(a) $t_0 = 1$ s after it begins to fall

(b) $t_0 = 7.4$ s, just before it hits the Colorado River

(c) At any time t_0.

Solution

(a) Use the definition of instantaneous velocity with $f(t) = 16t^2$ and $t_0 = 1$.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t \to 1} \frac{16t^2 - 16}{t - 1} = \lim_{t \to 1} \frac{16(t^2 - 1)}{t - 1}$$
$$= \lim_{t \to 1} \frac{16(t-1)(t+1)}{t - 1} = \lim_{t \to 1} [16(t+1)] = 32$$

At 1 s, the velocity of the rock is 32 ft/s.

(b) For $t_0 = 7.4$ s,

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to 7.4} \frac{f(t) - f(7.4)}{t - 7.4} = \lim_{t \to 7.4} \frac{16t^2 - 16 \cdot (7.4)^2}{t - 7.4}$$
$$= \lim_{t \to 7.4} \frac{16[t^2 - (7.4)^2]}{t - 7.4} = \lim_{t \to 7.4} \frac{16(t - 7.4)(t + 7.4)}{t - 7.4}$$
$$= \lim_{t \to 7.4} [16(t + 7.4)] = 16(14.8) = 236.8$$

At 7.4 s, the velocity of the rock is 236.8 ft/s.

NOTE Did you know? 236.8 ft/s is more than 161 mi/h!

(c)
$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{16t^2 - 16t_0{}^2}{t - t_0} = \lim_{t \to t_0} \frac{16\,(t - t_0)\,(t + t_0)}{t - t_0}$$
$$= 16 \lim_{t \to t_0} (t + t_0) = 32t_0$$

At t_0 seconds, the velocity of the rock is $32t_0$ ft/s. ■

NOW WORK Problem 33.

4 Find the Derivative of a Function at a Number

Slope of a tangent line, rate of change of a function, and velocity are all found using the same limit,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The common underlying idea is the mathematical concept of *derivative*.

DEFINITION Derivative of a Function at a Number

If $y = f(x)$ is a function and c is in the domain of f, then the **derivative** of f at c, denoted by $f'(c)$, is the number

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

provided this limit exists.

EXAMPLE 6 Finding the Derivative of a Function at a Number

Find the derivative of $f(x) = 2x^2 - 3x - 2$ at $x = 2$. That is, find $f'(2)$.

Solution

Using the definition of the derivative, we have

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(2x^2 - 3x - 2) - 0}{x - 2} \qquad f(2) = 2 \cdot 4 - 3 \cdot 2 - 2 = 0$$

$$= \lim_{x \to 2} \frac{(x-2)(2x+1)}{x - 2}$$

$$= \lim_{x \to 2} (2x + 1) = 5$$

■

NOW WORK Problem 23 and AP® Practice Problems 2 and 6.

So far we have given three interpretations of the derivative:

- *Geometric interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$.
- *Rate of change of a function interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f at c.
- *Physical interpretation:* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0.

EXAMPLE 7 Finding an Equation of a Tangent Line

(a) Find the derivative of $f(x) = \sqrt{2x}$ at $x = 8$.

(b) Use the derivative $f'(8)$ to find an equation of the tangent line to the graph of f at the point $(8, 4)$.

Solution

(a) The derivative of f at 8 is

$$f'(8) = \lim_{x \to 8} \frac{f(x) - f(8)}{x - 8} \underset{\uparrow}{=} \lim_{x \to 8} \frac{\sqrt{2x} - 4}{x - 8} \underset{\uparrow}{=} \lim_{x \to 8} \frac{(\sqrt{2x} - 4)(\sqrt{2x} + 4)}{(x - 8)(\sqrt{2x} + 4)}$$

$f(8) = \sqrt{2 \cdot 8} = 4$ Rationalize the numerator.

$$= \lim_{x \to 8} \frac{2x - 16}{(x - 8)(\sqrt{2x} + 4)} = \lim_{x \to 8} \frac{2(x - 8)}{(x - 8)(\sqrt{2x} + 4)} = \lim_{x \to 8} \frac{2}{\sqrt{2x} + 4} = \frac{1}{4}$$

(b) The slope of the tangent line to the graph of f at the point $(8, 4)$ is $f'(8) = \frac{1}{4}$. Using the point-slope form of a line, we get

$$y - 4 = f'(8)(x - 8) \qquad y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{4}(x - 8) \qquad f'(8) = \frac{1}{4}$$

$$y = \frac{1}{4}x + 2$$

■

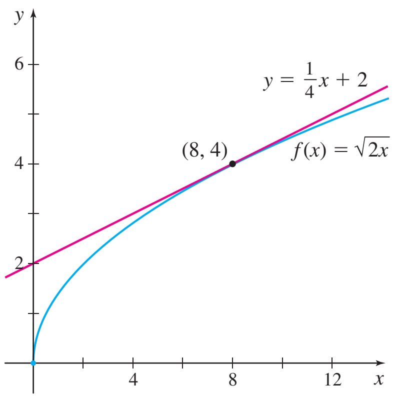


Figure 6

The graphs of f and the tangent line to the graph of f at $(8, 4)$ are shown in Figure 6.

NOW WORK Problem 15 and AP® Practice Problem 4.

EXAMPLE 8 Approximating the Derivative of a Function Defined by a Table

The table below lists several values of a function $y = f(x)$ that is continuous on the interval $[-1, 5]$ and has a derivative at each number in the interval $(-1, 5)$. Approximate the derivative of f at 2.

x	0	1	2	3	4
$f(x)$	0	3	12	33	72

Solution

There are several ways to approximate the derivative of a function defined by a table. Each uses an average rate of change to approximate the rate of change of f at 2, which is the derivative of f at 2.

- Using the average rate of change from 2 to 3, we have
$$\frac{f(3) - f(2)}{3 - 2} = \frac{33 - 12}{1} = 21$$
With this choice, $f'(2)$ is approximately 21.
- Using the average rate of change from 1 to 2, we have
$$\frac{f(2) - f(1)}{2 - 1} = \frac{12 - 3}{1} = 9$$
With this choice, $f'(2)$ is approximately 9.
- A third approximation can be found by averaging the above two approximations. Then $f'(2)$ is approximately $\dfrac{21 + 9}{2} = 15$. ■

NOW WORK Problem 51 and AP® Practice Problem 8.

2.1 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative is used to find instantaneous velocity.
2. *True or False* The derivative can be used to find the rate of change of a function.
3. The notation $f'(c)$ is read f ____ of c; $f'(c)$ represents the ____ of the tangent line to the graph of f at the point ____ .
4. *True or False* If it exists, $\lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$ is the derivative of the function f at 3.
5. If $f(x) = 6x - 3$, then $f'(3) =$ ________.
6. The velocity of an object, the slope of a tangent line, and the rate of change of a function are three different interpretations of the mathematical concept called the ________.

Skill Building

In Problems 7–16,

(a) Find an equation for the tangent line to the graph of each function at the indicated point.
(b) Find an equation of the normal line to each function at the indicated point.
(c) Graph the function, the tangent line, and the normal line at the indicated point on the same set of coordinate axes.

7. $f(x) = 3x^2$ at $(-2, 12)$
8. $f(x) = x^2 + 2$ at $(-1, 3)$
9. $f(x) = x^3$ at $(-2, -8)$
10. $f(x) = x^3 + 1$ at $(1, 2)$
11. $f(x) = \dfrac{1}{x}$ at $(1, 1)$ PAGE 163
12. $f(x) = \sqrt{x}$ at $(4, 2)$
13. $f(x) = \dfrac{1}{x + 5}$ at $\left(1, \dfrac{1}{6}\right)$
14. $f(x) = \dfrac{2}{x + 4}$ at $\left(1, \dfrac{2}{5}\right)$
15. $f(x) = \dfrac{1}{\sqrt{x}}$ at $(1, 1)$ PAGE 167
16. $f(x) = \dfrac{1}{x^2}$ at $(1, 1)$

In Problems 17–20, find the rate of change of f at the indicated numbers.

17. $f(x) = 5x - 2$ at **(a)** $c = 0$, **(b)** $c = 2$ PAGE 164
18. $f(x) = x^2 - 1$ at **(a)** $c = -1$, **(b)** $c = 1$
19. $f(x) = \dfrac{x^2}{x + 3}$ at **(a)** $c = 0$, **(b)** $c = 1$
20. $f(x) = \dfrac{x}{x^2 - 1}$ at **(a)** $c = 0$, **(b)** $c = 2$

In Problems 21–30, find the derivative of each function at the given number.

21. $f(x) = 2x + 3$ at 1
22. $f(x) = 3x - 5$ at 2
23. $f(x) = x^2 - 2$ at 0 PAGE 167
24. $f(x) = 2x^2 + 4$ at 1
25. $f(x) = 3x^2 + x + 5$ at -1
26. $f(x) = 2x^2 - x - 7$ at -1
27. $f(x) = \sqrt{x}$ at 4
28. $f(x) = \dfrac{1}{x^2}$ at 2
29. $f(x) = \dfrac{2 - 5x}{1 + x}$ at 0
30. $f(x) = \dfrac{2 + 3x}{2 + x}$ at 1

PAGE 166 **31. Approximating Velocity** An object in rectilinear motion moves according to the position function $s(t) = 10t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 3$ s by letting Δt first equal 0.1 s, then 0.01 s, and finally 0.001 s. What limit does the velocity appear to be approaching? Organize the results in a table.

32. Approximating Velocity An object in rectilinear motion moves according to the position function $s(t) = 5 - t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 1$ by letting Δt first equal 0.1, then 0.01, and finally 0.001. What limit does the velocity appear to be approaching? Organize the results in a table.

PAGE 166 **33. Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 3t^2 + 4t$. Find the velocity v at $t_0 = 0$. At $t_0 = 2$. At any time t_0.

34. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 2t^3 + 4$. Find the velocity v at $t_0 = 0$. At $t_0 = 3$. At any time t_0.

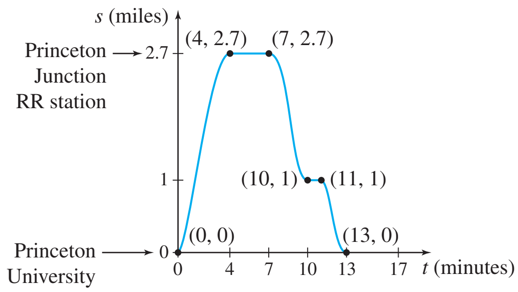
35. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 3t^2 - \dfrac{1}{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

36. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 2\sqrt{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

37. The Princeton Dinky is the shortest rail line in the country. It runs for 2.7 miles, connecting Princeton University to the Princeton Junction railroad station. The Dinky starts from the university and moves north toward Princeton Junction. Its distance from Princeton is shown in the graph (top, right), where the time t is in minutes and the distance s of the Dinky from Princeton University is in miles.

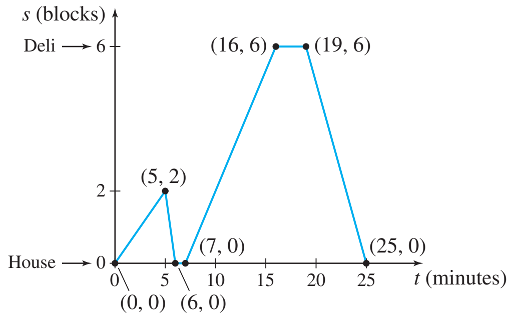
MancusoMichael/AP Images

(a) When is the Dinky headed toward Princeton University?
(b) When is it headed toward Princeton Junction?
(c) When is the Dinky stopped?
(d) Find its average velocity on a trip from Princeton to Princeton Junction.
(e) Find its average velocity for the round-trip shown in the graph, that is, from $t = 0$ to $t = 13$.



38. Barbara walks to the deli, which is six blocks east of her house. After walking two blocks, she realizes she left her phone on her desk, so she runs home. After getting the phone, and closing and locking the door, Barbara starts on her way again. At the deli, she waits in line to buy a bottle of **vitaminwater**™, and then she jogs home. The graph below represents Barbara's journey. The time t is in minutes, and s is Barbara's distance, in blocks, from home.

(a) At what times is she headed toward the deli?
(b) At what times is she headed home?
(c) When is the graph horizontal? What does this indicate?
(d) Find Barbara's average velocity from home until she starts back to get her phone.
(e) Find Barbara's average velocity from home to the deli after getting her phone.
(f) Find her average velocity from the deli to home.



Applications and Extensions

39. Slope of a Tangent Line An equation of the tangent line to the graph of a function f at (2, 6) is $y = -3x + 12$. What is $f'(2)$?

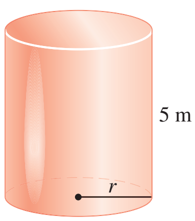
40. Slope of a Tangent Line An equation of the tangent line of a function f at (3, 2) is $y = \dfrac{1}{3}x + 1$. What is $f'(3)$?

41. Tangent Line Does the tangent line to the graph of $y = x^2$ at (1, 1) pass through the point (2, 5)?

42. Tangent Line Does the tangent line to the graph of $y = x^3$ at (1, 1) pass through the point (2, 5)?

PAGE 164 **43. Respiration Rate** A human being's respiration rate R (in breaths per minute) is given by $R = R(p) = 10.35 + 0.59p$, where p is the partial pressure of carbon dioxide in the lungs. Find the rate of change in respiration when $p = 50$.

44. Instantaneous Rate of Change The volume V of the right circular cylinder of height 5 m and radius r m shown in the figure is $V = V(r) = 5\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius when $r = 3$ m.



45. Market Share During a month-long advertising campaign, the total sales S of a magazine is modeled by the function $S(x) = 5x^2 + 100x + 10{,}000$, where x, $0 \le x \le 30$, represents the number of days since the campaign began.

(a) What is the average rate of change of sales from $x = 10$ to $x = 20$ days?

(b) What is the instantaneous rate of change of sales when $x = 10$ days?

46. Demand Equation The demand equation for an item is $p = p(x) = 90 - 0.02x$, where p is the price in dollars and x is the number of units (in thousands) made.

(a) Assuming all units made can be sold, find the revenue function $R(x) = xp(x)$.

(b) **Marginal Revenue** Marginal revenue is defined as the additional revenue earned by selling an additional unit. If we use $R'(x)$ to measure the marginal revenue, find the marginal revenue when 1 million units are sold.

47. Gravity If a ball is dropped from the top of the Empire State Building, 1002 ft above the ground, the distance s (in feet) it falls after t seconds is $s(t) = 16t^2$.

(a) What is the average velocity of the ball for the first 2 s?

(b) How long does it take for the ball to hit the ground?

(c) What is the average velocity of the ball during the time it is falling?

(d) What is the velocity of the ball when it hits the ground?

48. Velocity A ball is thrown upward. Its height h in feet is given by $h(t) = 100t - 16t^2$, where t is the time elapsed in seconds.

(a) What is the velocity v of the ball at $t = 0$ s, $t = 1$ s, and $t = 4$ s?

(b) At what time t does the ball strike the ground?

(c) At what time t does the ball reach its highest point? *Hint:* At the time the ball reaches its maximum height, it is stationary. So, its velocity $v = 0$.

49. Gravity A rock is dropped from a height of 88.2 m and falls toward Earth in a straight line. In t seconds the rock falls $4.9t^2$ m.

(a) What is the average velocity of the rock for the first 2 s?

(b) How long does it take for the rock to hit the ground?

(c) What is the average velocity of the rock during its fall?

(d) What is the velocity v of the rock when it hits the ground?

50. Velocity At a certain instant, the speedometer of an automobile reads V mi/h. During the next $\frac{1}{4}$ s the automobile travels 20 ft. Approximate V from this information.

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51. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 h the water is continuously used. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

(a) Use the table to approximate $A'(5)$.

(b) Using appropriate units, interpret $A'(5)$ in the context of the problem.

52. The table below lists the outside temperature T, in degrees Fahrenheit, in Naples, Florida, on a certain day in January, for selected times x, where x is the number of hours since 12 a.m.

x	5	7	9	11	12	13	14	16	17
$T(x)$	62	71	74	78	81	83	84	85	78

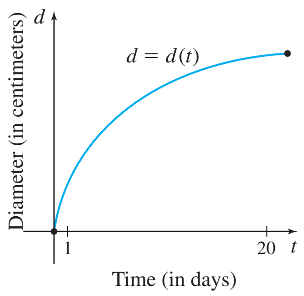
(a) Use the table to approximate $T'(11)$.

(b) Using appropriate units, interpret $T'(11)$ in the context of the problem.

53. Rate of Change Show that the rate of change of a linear function $f(x) = mx + b$ is the slope m of the line $y = mx + b$.

54. Rate of Change Show that the rate of change of a quadratic function $f(x) = ax^2 + bx + c$ is a linear function of x.

55. Agriculture The graph represents the diameter d (in centimeters) of a maturing peach as a function of the time t (in days) it is on the tree.

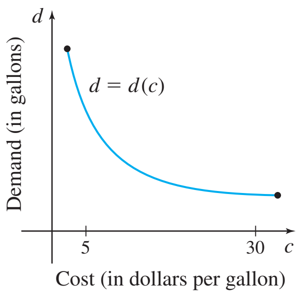


(a) Interpret the derivative $d'(t)$ as a rate of change.

(b) Which is larger, $d'(1)$ or $d'(20)$?

(c) Interpret both $d'(1)$ and $d'(20)$.

56. Business The graph represents the demand d (in gallons) for olive oil as a function of the cost c (in dollars per gallon) of the oil.



(a) Interpret the derivative $d'(c)$.

(b) Which is larger, $d'(5)$ or $d'(30)$? Give an interpretation to $d'(5)$ and $d'(30)$.

57. Volume of a Cube A metal cube with each edge of length x centimeters is expanding uniformly as a consequence of being heated.

(a) Find the average rate of change of the volume of the cube with respect to an edge as x increases from 2.00 to 2.01 cm.

(b) Find the instantaneous rate of change of the volume of the cube with respect to an edge at the instant when $x = 2$ cm.

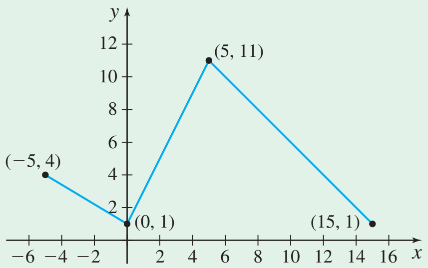
Preparing for the AP® Exam

AP® Practice Problems

PAGE 163
1. The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:

(A) $f(2) = 2$; $f'(2) = -1$ (B) $f(2) = 3$; $f'(2) = -1$
(C) $f(2) = 2$; $f'(2) = 1$ (D) $f(2) = 3$; $f'(2) = 2$

PAGE 167
2. The graph of the function f, given below, consists of three line segments. Find $f'(3)$.



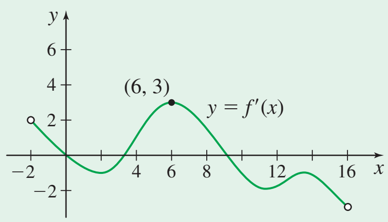
Graph of f

(A) 1 (B) 2 (C) 3 (D) $f'(3)$ does not exist

PAGE 164
3. What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?

(A) 5 (B) 7 (C) 12 (D) 17

PAGE 167
4. The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is

(A) $y = 3$ (B) $y + 2 = 6(x + 3)$
(C) $y + 2 = 6x$ (D) $y + 2 = 3(x - 6)$

PAGE 163
5. If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2, 6)$, then $f'(2) =$

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{13}{3}$

PAGE 167
6. If f is a function for which $\lim\limits_{x \to -3} \dfrac{f(x) - f(-3)}{x + 3} = 0$, then which of the following statements must be true?

(A) $x = -3$ is a vertical asymptote of the graph.
(B) The derivative of f at $x = -3$ exists.
(C) The function f is continuous at $x = 3$.
(D) f is not defined at $x = -3$.

PAGE 165
7. If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \le t \le 5$ is

(A) 5 (B) 20 (C) 40 (D) 100

PAGE 168
8. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

2.2 The Derivative as a Function; Differentiability

OBJECTIVES ***When you finish this section, you should be able to:***

1 **Define the derivative function (p. 171)**
2 **Graph the derivative function (p. 173)**
3 **Identify where a function is not differentiable (p. 175)**

1 Define the Derivative Function

The derivative of f at a real number c has been defined as the real number

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}} \tag{1}$$

provided the limit exists. We refer to this representation of the derivative as Form (1).

Another way to find the derivative of f at any real number is obtained by rewriting the expression $\dfrac{f(x)-f(c)}{x-c}$ and letting $x=c+h$, $h\neq 0$. Then

$$\frac{f(x)-f(c)}{x-c}=\frac{f(c+h)-f(c)}{(c+h)-c}=\frac{f(c+h)-f(c)}{h}$$

Since $x=c+h$, then as x approaches c, h approaches 0. Form (1) of the derivative with these changes becomes

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

So now, we have an equivalent way to write Form (1) for the derivative of f at a real number c.

$$\boxed{f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}}$$

See Figure 7.

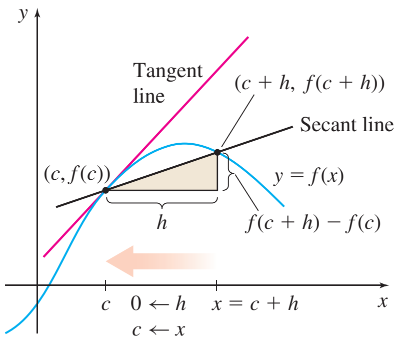


Figure 7 The slope of the tangent line at c is $f'(c)=\lim_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$

In the expression above for $f'(c)$, c is any real number. That is, the derivative f' is a function, called the *derivative function* of f. Now replace c by x, the independent variable of f.

DEFINITION The Derivative Function f'

The **derivative function** f' of a function f is

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} \qquad (2)$$

provided the limit exists. If f has a derivative, then f is said to be **differentiable**.

IN WORDS In Form (2) the derivative is the limit of a difference quotient.

We refer to this representation of the derivative as Form (2).

EXAMPLE 1 Finding the Derivative Function

Find the derivative of the function $f(x)=x^2-5x$ at any real number x using Form (2).

Solution

Using Form (2), we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^2-5(x+h)]-(x^2-5x)}{h}$$

$$=\lim_{h\to 0}\frac{[(x^2+2xh+h^2)-5x-5h]-x^2+5x}{h}=\lim_{h\to 0}\frac{2xh+h^2-5h}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h-5)}{h}=\lim_{h\to 0}(2x+h-5)=2x-5 \quad ■$$

NOTE Compare the solution and answer found in Example 1 to Example 2(b) on pp. 163–164.

NOW WORK AP® Practice Problem 2.

The domain of the function f' is the set of real numbers in the domain of f for which the limit (2) exists. So the domain of f' is a subset of the domain of f.

We can use either Form (1) or Form (2) to find derivatives. However, if we want the derivative of f at a specified number c, we usually use Form (1) to find $f'(c)$. If we want to find the derivative function of f, we usually use Form (2) to find $f'(x)$. In this section, we use the definitions of the derivative, Forms (1) and (2), to investigate derivatives. In the next section, we begin to develop formulas for finding the derivatives.

EXAMPLE 2 Finding the Derivative Function

NOTE The instruction "differentiate f" means "find the derivative of f."

Differentiate $f(x)=\sqrt{x}$ and determine the domain of f'.

Solution

The domain of f is $\{x|x\geq 0\}$. To find the derivative of f, we use Form (2). Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

We rationalize the numerator to find the limit.

$$f'(x)=\lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]=\lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

The limit does not exist when $x=0$. But for all other x in the domain of f, the limit does exist. So, the domain of the derivative function $f'(x)=\dfrac{1}{2\sqrt{x}}$ is $\{x|x>0\}$. ■

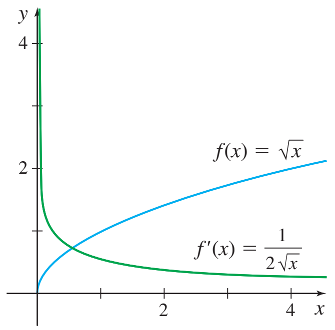


Figure 8

In Example 2, notice that the domain of the derivative function f' is a proper subset of the domain of the function f. The graphs of both f and f' are shown in Figure 8.

NOW WORK Problem 15.

EXAMPLE 3 Interpreting the Derivative as a Rate of Change

The surface area S (in square meters) of a balloon is expanding as a function of time t (in seconds) according to $S=S(t)=5t^2$. Find the rate of change of the surface area of the balloon with respect to time. What are the units of $S'(t)$?

Solution

An interpretation of the derivative function $S'(t)$ is the rate of change of $S=S(t)$.

$$S'(t)=\lim_{h\to 0}\frac{S(t+h)-S(t)}{h}=\lim_{h\to 0}\frac{5(t+h)^2-5t^2}{h}\qquad\text{Form (2)}$$

$$=\lim_{h\to 0}\frac{5(t^2+2th+h^2)-5t^2}{h}$$

$$=\lim_{h\to 0}\frac{5t^2+10th+5h^2-5t^2}{h}=\lim_{h\to 0}\frac{10th+5h^2}{h}$$

$$=\lim_{h\to 0}\frac{(10t+5h)h}{h}=\lim_{h\to 0}(10t+5h)=10t$$

Since $S'(t)$ is the limit of the quotient of a change in area divided by a change in time, the units of the rate of change are square meters per second (m^2/s). The rate of change of the surface area S of the balloon with respect to time is $10t$ m^2/s. ■

NOW WORK Problem 67 and AP® Practice Problems 10 and 11.

2 Graph the Derivative Function

There is a relationship between the graph of a function and the graph of its derivative.

EXAMPLE 4 Graphing a Function and Its Derivative

Find f' if $f(x)=x^3-1$. Then graph $y=f(x)$ and $y=f'(x)$ on the same set of coordinate axes.

Solution

$f(x)=x^3-1$ so

$$f(x+h)=(x+h)^3-1=x^3+3hx^2+3h^2x+h^3-1$$

Using Form (2), we find

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x^3+3hx^2+3h^2x+h^3-1)-(x^3-1)}{h}$$

$$=\lim_{h\to 0}\frac{3hx^2+3h^2x+h^3}{h}=\lim_{h\to 0}\frac{h(3x^2+3hx+h^2)}{h}$$

$$=\lim_{h\to 0}(3x^2+3hx+h^2)=3x^2$$

The graphs of f and f' are shown below in Figure 9. ■

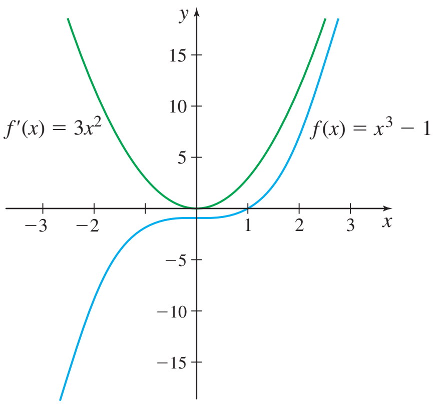


Figure 9

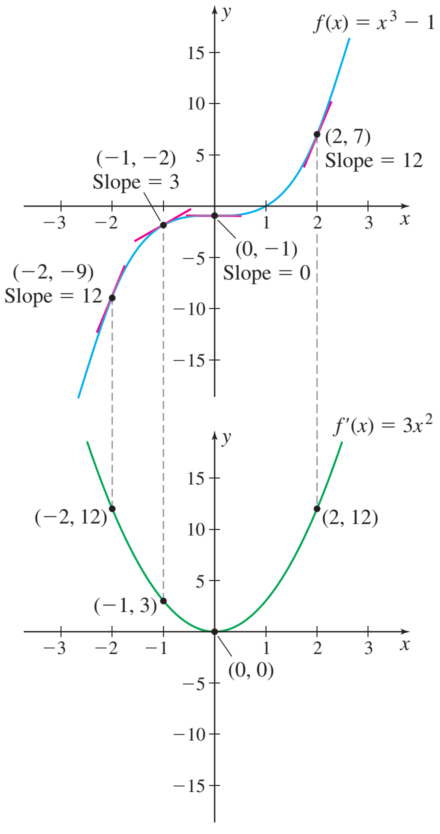


Figure 10

NOW WORK Problem 19.

Figure 10 illustrates several tangent lines to the graph of $f(x)=x^3-1$. Observe that the tangent line to the graph of f at $(0,-1)$ is horizontal, so its slope is 0. Then $f'(0)=0$, so the graph of f' contains the point $(0,0)$. Also notice that every tangent line to the graph of f has a nonnegative slope, so $f'(x)\geq 0$. That is, the range of the function f' is $\{y|y\geq 0\}$. Finally, notice that the slope of each tangent line is the y-coordinate of the corresponding point on the graph of the derivative f'.

With these ideas in mind, we can obtain a rough sketch of the derivative function f', even if we know only the graph of the function f.

EXAMPLE 5 Graphing the Derivative Function

Use the graph of the function $y=f(x)$, shown in Figure 11, to sketch the graph of the derivative function $y=f'(x)$.

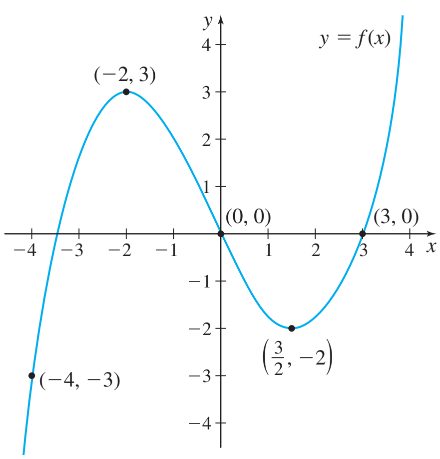


Figure 11

Solution

We begin by drawing tangent lines to the graph of f at the points shown in Figure 11. See the graph at the top of Figure 12. At the points $(-2,3)$ and $\left(\frac{3}{2},-2\right)$ the tangent lines are horizontal, so their slopes are 0. This means $f'(-2)=0$ and $f'\left(\frac{3}{2}\right)=0$, so the points $(-2,0)$ and $\left(\frac{3}{2},0\right)$ are on the graph of the derivative function at the bottom of Figure 12. Now we estimate the slope of the tangent lines at the other selected points.

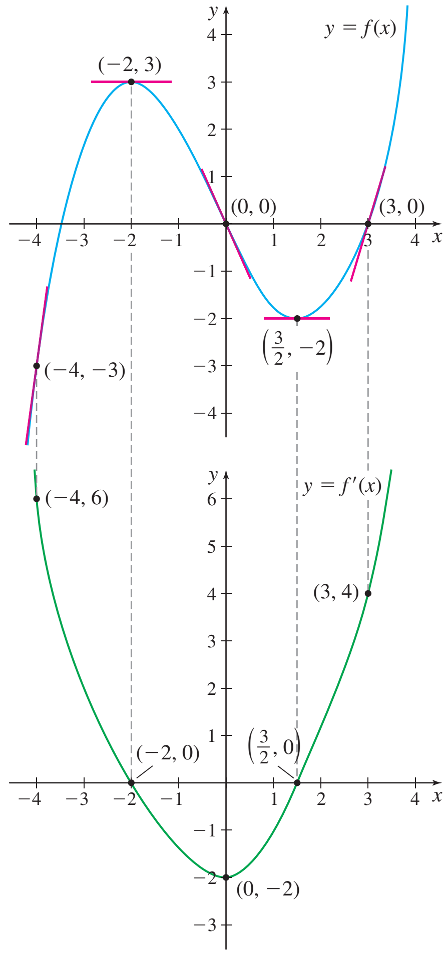


Figure 12

For example, at the point $(-4, -3)$, the slope of the tangent line is positive and the line is rather steep. We estimate the slope to be close to 6, and we plot the point $(-4, 6)$ on the bottom graph of Figure 12. Continue the process and then connect the points with a smooth curve. ■

Notice in Figure 12 that at the points on the graph of f where the tangent lines are horizontal, the graph of the derivative f' intersects the x-axis. Also notice that wherever the graph of f is increasing, the slopes of the tangent lines are positive, that is, f' is positive, so the graph of f' is above the x-axis. Similarly, wherever the graph of f is decreasing, the slopes of the tangent lines are negative, so the graph of f' is below the x-axis.

NOW WORK Problem 29.

3 Identify Where a Function Is Not Differentiable

Suppose a function f is continuous on an open interval containing the number c. The function f is not differentiable at the number c if $\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ does not exist. Three (of several) ways this can happen are:

- $\lim_{x\to c^-}\dfrac{f(x)-f(c)}{x-c}$ exists and $\lim_{x\to c^+}\dfrac{f(x)-f(c)}{x-c}$ exists, but they are not equal.*
  When this happens the graph of f has a **corner** at $(c, f(c))$. For example, the absolute value function $f(x)=|x|$ has a corner at $(0, 0)$. See Figure 13.
- The one-sided limits are both infinite and both equal ∞ or both equal $-\infty$.
  When this happens, the graph of f has a vertical tangent line at $(c, f(c))$.
  For example, the cube root function $f(x)=\sqrt[3]{x}$ has a vertical tangent at $(0, 0)$.
  See Figure 14.
- Both one-sided limits are infinite, but one equals $-\infty$ and the other equals ∞.
  When this happens, the graph of f has a vertical tangent line at the point $(c, f(c))$.
  This point is referred to as a **cusp**. For example, the function $f(x)=x^{2/3}$ has a cusp at $(0, 0)$. See Figure 15.

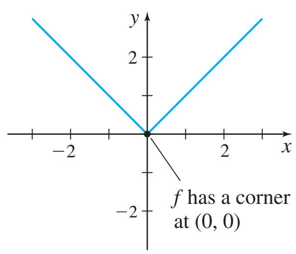


Figure 13 $f(x)=|x|$; $f'(0)$ does not exist.

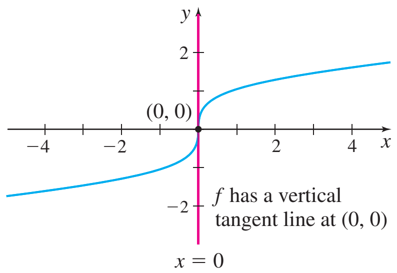


Figure 14 $f(x)=\sqrt[3]{x}$; $f'(0)$ does not exist.

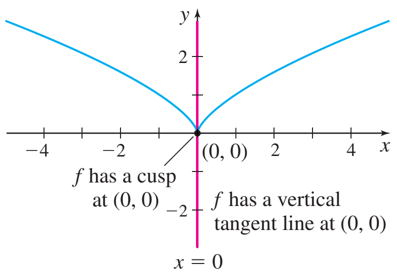


Figure 15 $f(x)=x^{2/3}$; $f'(0)$ does not exist.

EXAMPLE 6 Identifying Where a Function Is Not Differentiable

Given the piecewise defined function $f(x)=\begin{cases}-2x^2+4 & \text{if } x<1\\ x^2+1 & \text{if } x\ge 1\end{cases}$, determine whether $f'(1)$ exists.

Solution

Use Form (1) of the definition of a derivative to determine whether $f'(1)$ exists.

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{f(x)-2}{x-1}\qquad f(1)=1^2+1=2$$

---

*The one-sided limits, $\lim_{x\to c^-}\dfrac{f(x)-f(c)}{x-c}$ and $\lim_{x\to c^+}\dfrac{f(x)-f(c)}{x-c}$, are called the **left-hand derivative** of f at c, denoted $f'_-(c)$, and the **right-hand derivative** of f at c, denoted $f'_+(c)$, respectively. Using properties of limits, the derivative $f'(c)$ exists if and only if $f'_-(c)=f'_+(c)$.

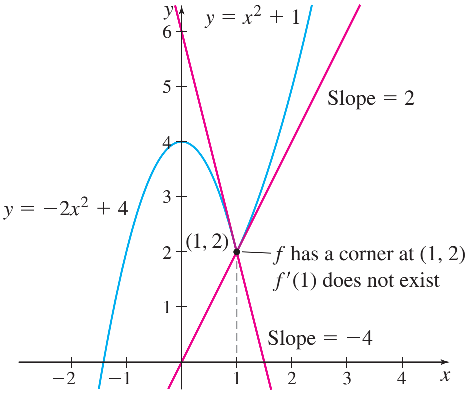


Figure 16 f has a corner at (1, 2).

If $x < 1$, then $f(x) = -2x^2 + 4$; if $x \geq 1$, then $f(x) = x^2 + 1$. So, it is necessary to find the one-sided limits at 1.

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(-2x^2 + 4) - 2}{x - 1} = \lim_{x \to 1^-} \frac{-2(x^2 - 1)}{x - 1}$$
$$= -2 \lim_{x \to 1^-} \frac{(x - 1)(x + 1)}{x - 1} = -2 \lim_{x \to 1^-} (x + 1) = -4$$
$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x^2 + 1) - 2}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2$$

Since the one-sided limits are not equal, $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ does not exist, and so $f'(1)$ does not exist. ■

Figure 16 illustrates the graph of the function f from Example 6. At 1, where the derivative does not exist, the graph of f has a corner. We usually say that the graph of f is not *smooth* at a corner.

NOW WORK Problem 39 and AP® Practice Problems 1, 5 and 9.

Example 7 illustrates the behavior of the graph of a function f when the derivative at a number c does not exist because $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ is infinite.

CALC CLIP

EXAMPLE 7 Showing That a Function Is Not Differentiable

Show that $f(x) = (x - 2)^{4/5}$ is not differentiable at 2.

Solution

The function f is continuous for all real numbers and $f(2) = (2 - 2)^{4/5} = 0$. Use Form (1) of the definition of the derivative to find the two one-sided limits at 2.

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(x - 2)^{4/5} - 0}{x - 2} = \lim_{x \to 2^-} \frac{(x - 2)^{4/5}}{x - 2} = \lim_{x \to 2^-} \frac{1}{(x - 2)^{1/5}} = -\infty$$
$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(x - 2)^{4/5} - 0}{x - 2} = \lim_{x \to 2^+} \frac{(x - 2)^{4/5}}{x - 2} = \lim_{x \to 2^+} \frac{1}{(x - 2)^{1/5}} = \infty$$

Since $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = -\infty$ and $\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \infty$, we conclude that the function f is not differentiable at 2. The graph of f has a vertical tangent line at the point (2, 0), which is a cusp of the graph. See Figure 17. ■

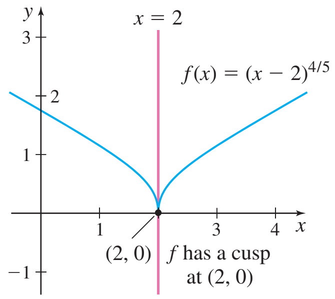


Figure 17 $f'(2)$ does not exist; the point (2, 0) is a cusp of the graph of f.

NOW WORK Problem 35.

EXAMPLE 8 Obtaining Information about $y = f(x)$ from the Graph of Its Derivative Function

Suppose $y = f(x)$ is continuous for all real numbers. Figure 18 shows the graph of its derivative function f'.

(a) Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.

(b) Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.

(c) Does the graph of f have any corners? If yes, explain why and identify where they occur.

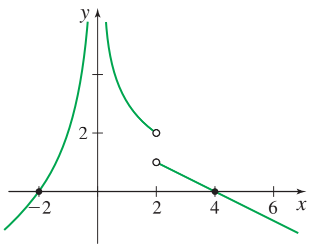


Figure 18 $y = f'(x)$

Solution

(a) Since the derivative f' equals the slope of a tangent line, horizontal tangent lines occur where the derivative equals 0. Since $f'(x) = 0$ for $x = -2$ and $x = 4$, the graph of f has two horizontal tangent lines, one at the point $(-2, f(-2))$ and the other at the point $(4, f(4))$.

(b) As x approaches 0, the derivative function f' approaches ∞ both for $x<0$ and for $x>0$. The graph of f has a vertical tangent line at $x=0$. The point $(0, f(0))$ is not a cusp because both limits equal ∞.

(c) The derivative is not defined at 2 but the one-sided derivatives have unequal finite limits as x approaches 2. So the graph of f has a corner at $(2, f(2))$. ■

NOW WORK Problem 45.

Differentiability and Continuity

In Chapter 1, we investigated the continuity of a function. Here we have been investigating the differentiability of a function. An important connection exists between continuity and differentiability.

NEED TO REVIEW? Continuity is discussed in Section 1.3, pp. 102–110.

THEOREM

If a function f is differentiable at a number c, then f is continuous at c.

Proof To show that f is continuous at c, we need to verify that $\lim_{x\to c} f(x) = f(c)$. We begin by observing that if $x \neq c$, then

$$f(x)-f(c)=\left[\frac{f(x)-f(c)}{x-c}\right](x-c)$$

We take the limit of both sides as $x \to c$, and use the fact that the limit of a product equals the product of the limits (we show later that each limit exists).

$$\lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left\{\left[\frac{f(x)-f(c)}{x-c}\right](x-c)\right\}=\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\left[\lim_{x\to c}(x-c)\right]$$

Since f is differentiable at c, we know that

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)$$

is a number. Also for any real number c, $\lim_{x\to c}(x-c)=0$. So

$$\lim_{x\to c}[f(x)-f(c)]=[f'(c)]\left[\lim_{x\to c}(x-c)\right]=f'(c)\cdot 0=0$$

That is, $\lim_{x\to c} f(x)=f(c)$, so f is continuous at c. ■

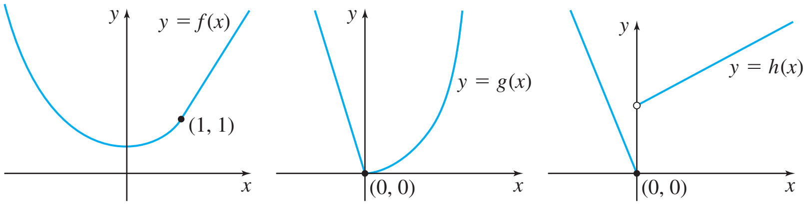
An equivalent statement of this theorem gives a condition under which a function has no derivative.

COROLLARY

If a function f is discontinuous at a number c, then f is not differentiable at c.

Let's look at some of the possibilities. In Figure 19(a), the function f is continuous at the number 1 and has a derivative at 1. The function g, graphed in Figure 19(b), is continuous at the number 0, but it has no derivative at 0. So continuity at a number c provides no prediction about differentiability. On the other hand, the function h graphed in Figure 19(c) illustrates the corollary: If h is discontinuous at a number, it is not differentiable at that number.

IN WORDS Differentiability implies continuity, but continuity does not imply differentiability.



(a) f is continuous at 1, and $f'(1)$ exists.

(b) g is continuous at 0, but $g'(0)$ does not exist.

(c) h is discontinuous at 0, so $h'(0)$ does not exist.

Figure 19

The corollary is useful if we are seeking the derivative of a function f that we suspect is discontinuous at a number c. If we can show that f is discontinuous at c, then the corollary affirms that the function f has no derivative at c. For example, since the floor function $f(x) = \lfloor x \rfloor$ is discontinuous at every integer c, it has no derivative at an integer.

EXAMPLE 9 Determining Whether a Function Is Differentiable at a Number

Determine whether the function

$$f(x) = \begin{cases} 2x+2 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ x^2 - 1 & \text{if } x > 3 \end{cases}$$

is differentiable at 3.

Solution

Since f is a piecewise-defined function, it may be discontinuous at 3 and therefore not differentiable at 3. So we begin by determining whether f is continuous at 3.

Since $f(3) = 5$, the function f is defined at 3. Use one-sided limits to check whether $\lim\limits_{x\to 3} f(x)$ exists.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2x+2) = 8 \qquad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2-1) = 8$$

Since $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$, then $\lim\limits_{x\to 3} f(x)$ exists. But $\lim\limits_{x\to 3} f(x) = 8$ and $f(3) = 5$, so f is discontinuous at 3. From the corollary, since f is discontinuous at 3, the function f is not differentiable at 3. ■

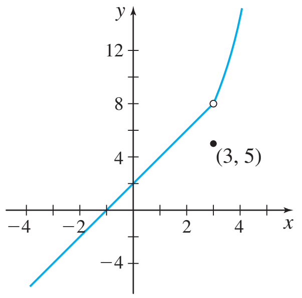


Figure 20 $f(x) = \begin{cases} 2x+2 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ x^2 - 1 & \text{if } x > 3 \end{cases}$

Figure 20 shows the graph of f.

NOW WORK Problem 43.

In Example 9, the function f is discontinuous at 3, so by the corollary, the derivative of f at 3 does not exist. But when a function is continuous at a number c, then sometimes the derivative at c exists and other times the derivative at c does not exist.

EXAMPLE 10 Determining Whether a Function Is Differentiable at a Number

Determine whether each piecewise-defined function is differentiable at c. If the function has a derivative at c, find it.

(a) $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases} \quad c = 0$ **(b)** $g(x) = \begin{cases} 1-2x & \text{if } x \le 1 \\ x-2 & \text{if } x > 1 \end{cases} \quad c = 1$

Solution

(a) See Figure 21. The function f is continuous at 0, which you should verify. To determine whether f has a derivative at 0, we examine the one-sided limits at 0 using Form (1).

For $x < 0$,

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^3 - 0}{x} = \lim_{x\to 0^-} x^2 = 0$$

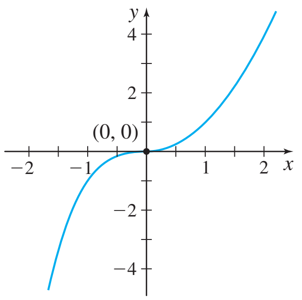


Figure 21 $f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

For $x > 0$,

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2 - 0}{x} = \lim_{x\to 0^+} x = 0$$

Since both one-sided limits are equal, f is differentiable at 0, and $f'(0) = 0$.

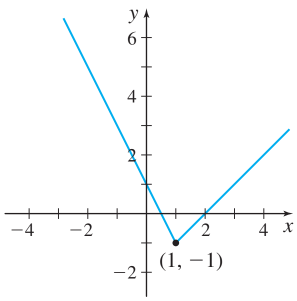


Figure 22 $g(x) = \begin{cases} 1-2x & \text{if } x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$

(**b**) See Figure 22. The function g is continuous at 1, which you should verify. To determine whether g is differentiable at 1, examine the one-sided limits at 1 using Form (1).

For $x < 1$,

$$\lim_{x\to1^-}\frac{g(x)-g(1)}{x-1} = \lim_{x\to1^-}\frac{(1-2x)-(-1)}{x-1} = \lim_{x\to1^-}\frac{2-2x}{x-1}$$
$$= \lim_{x\to1^-}\frac{-2(x-1)}{x-1} = \lim_{x\to1^-}(-2) = -2$$

For $x > 1$,

$$\lim_{x\to1^+}\frac{g(x)-g(1)}{x-1} = \lim_{x\to1^+}\frac{(x-2)-(-1)}{x-1} = \lim_{x\to1^+}\frac{x-1}{x-1} = 1$$

The one-sided limits are not equal, so $\lim_{x\to1}\frac{g(x)-g(1)}{x-1}$ does not exist. That is, g is not differentiable at 1. ■

Notice in Figure 21 the tangent lines to the graph of f turn smoothly around the origin. On the other hand, notice in Figure 22 the tangent lines to the graph of g change abruptly at the point $(1, -1)$, where the graph of g has a corner.

NOW WORK Problem 41 and AP® Practice Problems 3, 4, 6, and 7.

2.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The domain of a function f and the domain of its derivative function f' are always equal.
2. *True or False* If a function is continuous at a number c, then it is differentiable at c.
3. *Multiple Choice* If f is continuous at a number c and if $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ is infinite, then the graph of f has

   [(**a**) a horizontal (**b**) a vertical (**c**) no]

   tangent line at c.
4. The instruction, "Differentiate f," means to find the ____ of f.

Skill Building

In Problems 5–10, find the derivative of each function f at any real number c. Use Form (1) on page 171.

5. $f(x) = 10$
6. $f(x) = -4$
7. $f(x) = 2x + 3$
8. $f(x) = 3x - 5$
9. $f(x) = 2 - x^2$
10. $f(x) = 2x^2 + 4$

In Problems 11–16, differentiate each function f and determine the domain of f'. Use Form (2) on page 172.

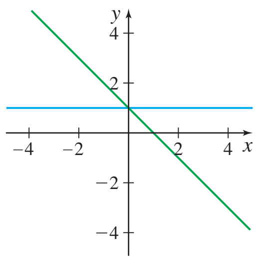
11. $f(x) = 5$
12. $f(x) = -2$
13. $f(x) = 3x^2 + x + 5$
14. $f(x) = 2x^2 - x - 7$
15. $f(x) = 5\sqrt{x-1}$ PAGE 173
16. $f(x) = 4\sqrt{x+3}$

In Problems 17–22, differentiate each function f. Graph $y = f(x)$ and $y = f'(x)$ on the same set of coordinate axes.

17. $f(x) = \frac{1}{3}x + 1$
18. $f(x) = -4x - 5$
19. $f(x) = 2x^2 - 5x$ PAGE 174
20. $f(x) = -3x^2 + 2$
21. $f(x) = x^3 - 8x$
22. $f(x) = -x^3 - 8$

In Problems 23–26, for each figure determine if the graphs represent a function f and its derivative f'. If they do, indicate which is the graph of f and which is the graph of f'.

23.



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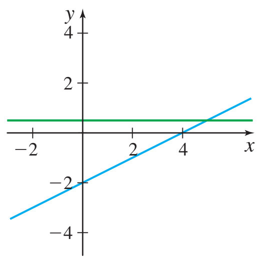

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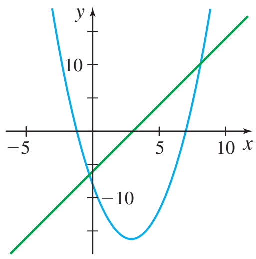

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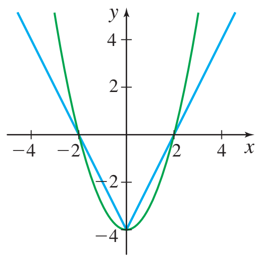
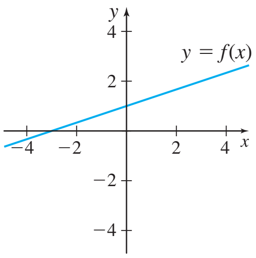


In Problems 27–30, use the graph of f to obtain the graph of f'.

27.



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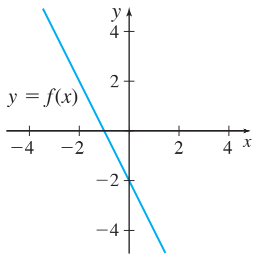
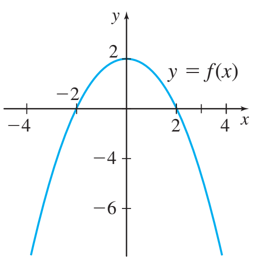


PAGE 175 **29.**



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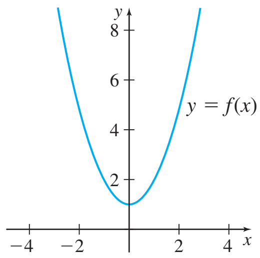
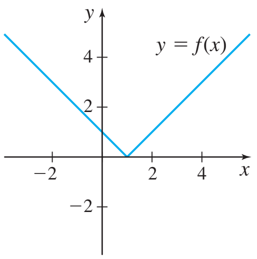


In Problems 31–34, the graph of a function f is given. Match each graph to the graph of its derivative f' in A–D.

31.



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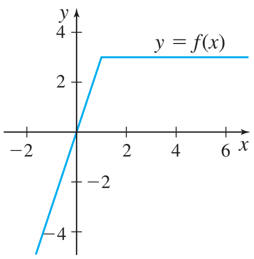

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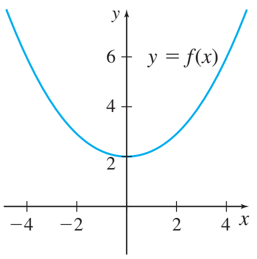

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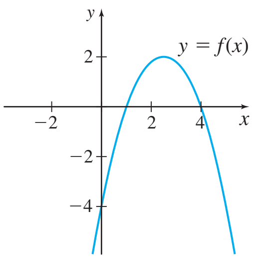
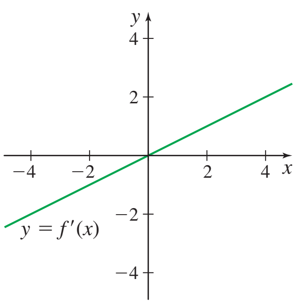
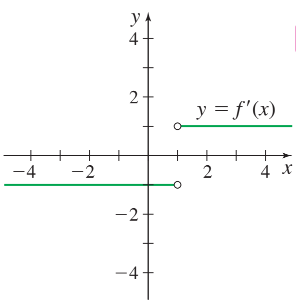
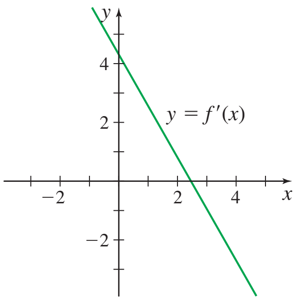




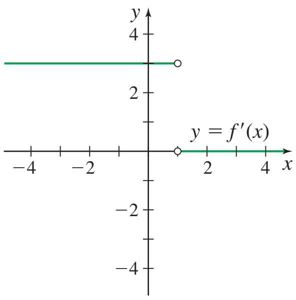
(A)



(B)



(C)



(D)

In Problems 35–44, determine whether each function f has a derivative at c. If it does, what is $f'(c)$? If it does not, give the reason why.

PAGE 176 **35.** $f(x)=x^{2/3}$ at $c=-8$

36. $f(x)=2x^{1/3}$ at $c=0$

37. $f(x)=|x^2-4|$ at $c=2$

38. $f(x)=|x^2-4|$ at $c=-2$

PAGE 176 **39.** $f(x)=\begin{cases} 2x+3 & \text{if } x<1 \\ x^2+4 & \text{if } x\geq 1 \end{cases}$ at $c=1$

40. $f(x)=\begin{cases} 3-4x & \text{if } x<-1 \\ 2x+9 & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 179 **41.** $f(x)=\begin{cases} -4+2x & \text{if } x\leq \frac{1}{2} \\ 4x^2-4 & \text{if } x>\frac{1}{2} \end{cases}$ at $c=\frac{1}{2}$

42. $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ -1-4x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 178 **43.** $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ 2+2x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

44. $f(x)=\begin{cases} 5-2x & \text{if } x<2 \\ x^2 & \text{if } x\geq 2 \end{cases}$ at $c=2$

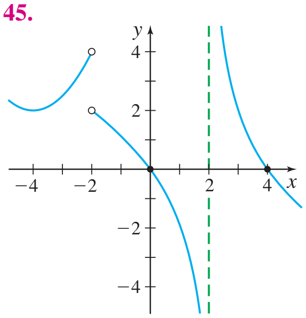
In Problems 45–48, each function f is continuous for all real numbers, and the graph of $y=f'(x)$ is given.

(a) *Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.*

(b) *Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.*

(c) *Does the graph of f have any corners? If yes, explain why and identify where they occur.*

PAGE 177 **45.**



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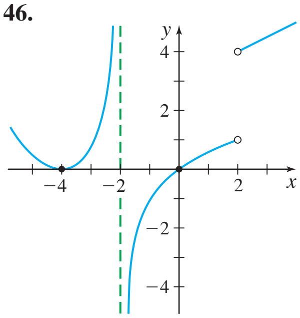

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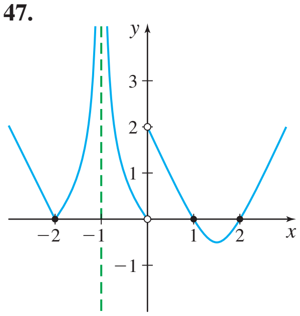

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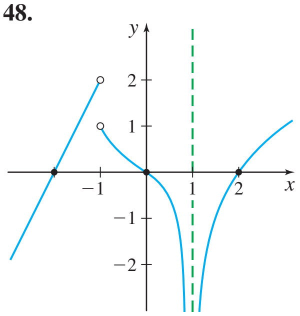
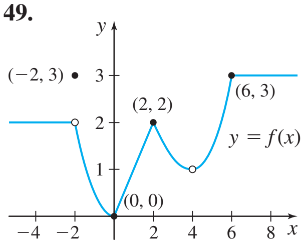


In Problems 49 and 50, use the given points $(c, f(c))$ on the graph of the function f.

(a) *For which numbers c does $\lim_{x \to c} f(x)$ exist but f is not continuous at c?*

(b) *For which numbers c is f continuous at c but not differentiable at c?*

49.



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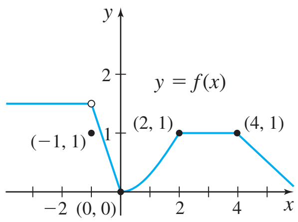


In Problems 51–54, find the derivative of each function.

51. $f(x) = mx + b$

52. $f(x) = ax^2 + bx + c$

53. $f(x) = \dfrac{1}{x^2}$

54. $f(x) = \dfrac{1}{\sqrt{x}}$

Applications and Extensions

In Problems 55–66, each limit represents the derivative of a function f at some number c. Determine f and c in each case.

55. $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$

56. $\lim_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$

57. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

58. $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$

59. $\lim_{h \to 0} \dfrac{\sqrt{9+h} - 3}{h}$

60. $\lim_{h \to 0} \dfrac{(8+h)^{1/3} - 2}{h}$

61. $\lim_{x \to \pi/6} \dfrac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}}$

62. $\lim_{x \to \pi/4} \dfrac{\cos x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

63. $\lim_{x \to 0} \dfrac{2(x+2)^2 - (x+2) - 6}{x}$

64. $\lim_{x \to 0} \dfrac{3x^3 - 2x}{x}$

65. $\lim_{h \to 0} \dfrac{(3+h)^2 + 2(3+h) - 15}{h}$

66. $\lim_{h \to 0} \dfrac{3(h-1)^2 + h - 3}{h}$

PAGE 173 **67. Units** The volume V (in cubic feet) of a balloon is expanding according to $V = V(t) = 4t$, where t is the time (in seconds). Find the rate of change of the volume of the balloon with respect to time. What are the units of $V'(t)$?

68. Units The area A (in square miles) of a circular patch of oil is expanding according to $A = A(t) = 2t$, where t is the time (in hours). At what rate is the area changing with respect to time? What are the units of $A'(t)$?

69. Units A manufacturer of precision digital switches has a daily cost C (in dollars) of $C(x) = 10{,}000 + 3x$, where x is the number of switches produced daily. What is the rate of change of cost with respect to x? What are the units of $C'(x)$?

70. Units A manufacturer of precision digital switches has daily revenue R (in dollars) of $R(x) = 5x - \dfrac{x^2}{2000}$, where x is the number of switches produced daily. What is the rate of change of revenue with respect to x? What are the units of $R'(x)$?

71. $f(x) = \begin{cases} x^3 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.

(b) Determine whether $f'(0)$ exists.

(c) Graph the function f and its derivative f'.

72. For the function $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.

(b) Determine whether $f'(0)$ exists.

(c) Graph the function f and its derivative f'.

73. Velocity The distance s (in feet) of an automobile from the origin at time t (in seconds) is given by the position function

$$s = s(t) = \begin{cases} t^3 & \text{if } 0 \le t < 5 \\ 125 & \text{if } t \ge 5 \end{cases}$$

(This could represent a crash test in which a vehicle is accelerated until it hits a brick wall at $t = 5$ s.)

(a) Find the velocity just before impact (at $t = 4.99$ s) and just after impact (at $t = 5.01$ s).

(b) Is the velocity function $v = s'(t)$ continuous at $t = 5$?

(c) How do you interpret the answer to (b)?

74. Population Growth A simple model for population growth states that the rate of change of population size P with respect to time t is proportional to the population size. Express this statement as an equation involving a derivative.

75. Atmospheric Pressure Atmospheric pressure p decreases as the distance x from the surface of Earth increases, and the rate of change of pressure with respect to altitude is proportional to the pressure. Express this law as an equation involving a derivative.

76. Electrical Current Under certain conditions, an electric current I will die out at a rate (with respect to time t) that is proportional to the current remaining. Express this law as an equation involving a derivative.

77. Tangent Line Let $f(x) = x^2 + 2$. Find all points on the graph of f for which the tangent line passes through the origin.

78. Tangent Line Let $f(x) = x^2 - 2x + 1$. Find all points on the graph of f for which the tangent line passes through the point $(1, -1)$.

79. Area and Circumference of a Circle A circle of radius r has area $A=\pi r^2$ and circumference $C=2\pi r$. If the radius changes from r to $r+\Delta r$, find the:

(a) Change in area.
(b) Change in circumference.
(c) Average rate of change of area with respect to radius.
(d) Average rate of change of circumference with respect to radius.
(e) Rate of change of circumference with respect to radius.

80. Volume of a Sphere The volume V of a sphere of radius r is $V=\dfrac{4\pi r^3}{3}$. If the radius changes from r to $r+\Delta r$, find the:

(a) Change in volume.
(b) Average rate of change of volume with respect to radius.
(c) Rate of change of volume with respect to radius.

81. Use the definition of the derivative to show that $f(x)=|x|$ is not differentiable at 0.

82. Use the definition of the derivative to show that $f(x)=\sqrt[3]{x}$ is not differentiable at 0.

83. If f is an even function that is differentiable at c, show that its derivative function is odd. That is, show $f'(-c)=-f'(c)$.

84. If f is an odd function that is differentiable at c, show that its derivative function is even. That is, show $f'(-c)=f'(c)$.

85. Tangent Lines and Derivatives Let f and g be two functions, each with derivatives at c. State the relationship between their tangent lines at c if:

(a) $f'(c)=g'(c)$ **(b)** $f'(c)=-\dfrac{1}{g'(c)}$ $g'(c)\neq 0$

Challenge Problems

86. Let f be a function defined for all real numbers x. Suppose f has the following properties:

$$f(u+v)=f(u)f(v) \qquad f(0)=1 \qquad f'(0) \text{ exists}$$

(a) Show that $f'(x)$ exists for all real numbers x.
(b) Show that $f'(x)=f'(0)f(x)$.

87. A function f is defined for all real numbers and has the following three properties:

$$f(1)=5 \qquad f(3)=21 \qquad f(a+b)-f(a)=kab+2b^2$$

for all real numbers a and b where k is a fixed real number independent of a and b.

(a) Use $a=1$ and $b=2$ to find k.
(b) Find $f'(3)$.
(c) Find $f'(x)$ for all real x.

88. A function f is **periodic** if there is a positive number p so that $f(x+p)=f(x)$ for all x. Suppose f is differentiable. Show that if f is periodic with period p, then f' is also periodic with period p.

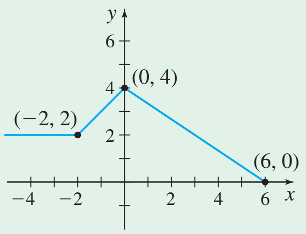
Preparing for the AP® Exam

AP® Practice Problems

PAGE 176
1. The function $f(x)=\begin{cases} x^2-ax & \text{if } x\le 1 \\ ax+b & \text{if } x>1\end{cases}$, where a and b are constants. If f is differentiable at $x=1$, then $a+b=$

(A) -3 (B) -2 (C) 0 (D) 2

PAGE 172
2. The graph of the function f, given below, consists of three line segments. Find $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$.



(A) -1 (B) $-\dfrac{2}{3}$ (C) $-\dfrac{3}{2}$ (D) does not exist

PAGE 179
3. If $f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & \text{if } x\neq 5\\ 5 & \text{if } x=5\end{cases}$

which of the following statements about f are true?

I. $\lim\limits_{x\to 5} f$ exists.
II. f is continuous at $x=5$.
III. f is differentiable at $x=5$.

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 179
4. Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0)=3$, $f(2)=-3$, and $f(7)=3$, which of the following must be true?

I. f has at least 2 zeros.
II. f is continuous on the closed interval $[-1, 7]$.
III. For some c, $0<c<7$, $f(c)=-2$.

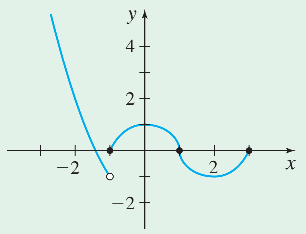
(A) I only (B) I and II only
(C) II and III only (D) I, II, and III

PAGE 176
5. If $f(x)=|x|$, which of the following statements about f are true?

I. f is continuous at 0.
II. f is differentiable at 0.
III. $f(0)=0$.

(A) I only (B) III only
(C) I and III only (D) I, II, and III

PAGE 179 **6.** The graph of the function f shown in the figure has horizontal tangent lines at the points (0, 1) and (2, −1) and a vertical tangent line at the point (1, 0). For what numbers x in the open interval (−2, 3) is f not differentiable?



(A) −1 only (B) −1 and 1 only

(C) −1, 0, and 2 only (D) −1, 0, 1, and 2

PAGE 179 **7.** Let f be a function for which $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = -3$. Which of the following must be true?

I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only

(C) I and II only (D) I, II, and III

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

(A) (1, −3) (B) (1, 2) (C) (2, 0) (D) (2, 4)

PAGE 176 **9.** At $x = 2$, the function $f(x) = \begin{cases} 4x+1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
(B) Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

PAGE 173 **10.** Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.
(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

PAGE 173 **11.** A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.
(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

OBJECTIVES *When you finish this section, you should be able to:*

1. **Differentiate a constant function (p. 184)**
2. **Differentiate a power function (p. 184)**
3. **Differentiate the sum and the difference of two functions (p. 186)**
4. **Differentiate the exponential function $y = e^x$ (p. 189)**

Finding the derivative of a function from the definition can become tedious, especially if the function f is complicated. Just as we did for limits, we derive some basic derivative formulas and some properties of derivatives that make finding a derivative simpler.

Before getting started, we introduce other notations commonly used for the derivative $f'(x)$ of a function $y = f(x)$. The most common ones are

$$y' \qquad \frac{dy}{dx} \qquad Df(x)$$

Leibniz notation $\dfrac{dy}{dx}$ may be written in several equivalent ways as

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

where $\dfrac{d}{dx}$ is an instruction to find the derivative (with respect to the independent variable x) of the function $y = f(x)$.

In **operator notation** $Df(x)$, D is said to *operate* on the function, and the result is the derivative of f. To emphasize that the operation is performed with respect to the independent variable x, it is sometimes written $Df(x) = D_x f(x)$.

We use prime notation or Leibniz notation, or sometimes a mixture of the two, depending on which is more convenient. We do not use the notation $Df(x)$ in this book.

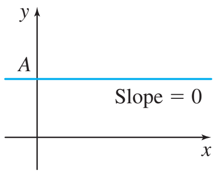


Figure 23 $f(x) = A$

1 Differentiate a Constant Function

See Figure 23. Since the graph of a constant function $f(x) = A$ is a horizontal line, the tangent line to f at any point is also a horizontal line, whose slope is 0. Since the derivative is the slope of the tangent line, the derivative of f is 0.

THEOREM Derivative of a Constant Function

If f is the constant function $f(x) = A$, then

$$\boxed{f'(x) = 0}$$

That is, if A is a constant, then

$$\boxed{\frac{d}{dx}A = 0}$$

IN WORDS The derivative of a constant is 0.

Proof If $f(x) = A$, then its derivative function is given by

$$f'(x) = \underbrace{\lim_{h\to 0}}_{\text{The definition of a derivative, Form (3)}} \frac{f(x+h) - f(x)}{h} = \underbrace{\lim_{h\to 0}}_{f(x) = A,\ f(x+h) = A} \frac{A - A}{h} = 0$$

■

EXAMPLE 1 Differentiating a Constant Function

(a) If $f(x) = \sqrt{3}$, then $f'(x) = 0$ **(b)** If $f(x) = -\frac{1}{2}$, then $f'(x) = 0$

(c) If $f(x) = \pi$, then $\frac{d}{dx}\pi = 0$ **(d)** If $f(x) = 0$, then $\frac{d}{dx}0 = 0$ ■

2 Differentiate a Power Function

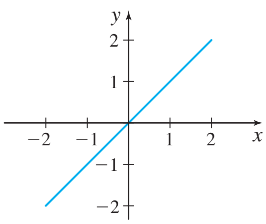


Figure 24 $f(x) = x$

Next we analyze the derivative of a power function $f(x) = x^n$, where $n \geq 1$ is an integer.

When $n = 1$, then $f(x) = x$ is the identity function and its graph is the line $y = x$, as shown in Figure 24.

The slope of the line $y = x$ is 1, so we would expect $f'(x) = 1$.

Proof $$f'(x) = \frac{d}{dx}x = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \underbrace{\lim_{h\to 0}}_{f(x) = x,\ f(x+h) = x+h}\frac{(x+h) - x}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

■

THEOREM Derivative of $f(x) = x$

If $f(x) = x$, then

$$\boxed{f'(x) = \frac{d}{dx}x = 1}$$

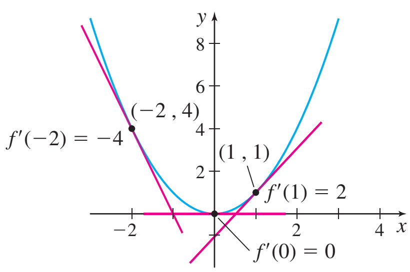


Figure 25 $f(x) = x^2$

When $n = 2$, then $f(x) = x^2$ is the square function. The derivative of f is

$$f'(x) = \frac{d}{dx}x^2 = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

The slope of the tangent line to the graph of $f(x) = x^2$ is different for every number x. Figure 25 shows the graph of f and several of its tangent lines. Notice that the slope of each tangent line drawn is twice the value of x.

When $n = 3$, then $f(x) = x^3$ is the cube function. The derivative of f is

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

Notice that the derivative of each of these power functions is another power function, whose degree is 1 less than the degree of the original function and whose coefficient is the degree of the original function. This rule holds for all power functions as the following theorem, called the *Simple Power Rule*, indicates.

IN WORDS The derivative of x raised to an integer power $n \geq 1$ is n times x raised to the power $n - 1$.

THEOREM Simple Power Rule

The derivative of the power function $y = x^n$, where $n \geq 1$ is an integer, is

$$\boxed{y' = \frac{d}{dx} x^n = nx^{n-1}}$$

NEED TO REVIEW? The Binomial Theorem is discussed in Section P.8, pp. 72–73.

Proof If $f(x) = x^n$ and n is a positive integer, then $f(x+h) = (x+h)^n$. We use the Binomial Theorem to expand $(x+h)^n$. Then

$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{6}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n}{h}$$ **Simplify.**

$$= \lim_{h \to 0} \frac{h\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}\right]}{h}$$ **Factor h in the numerator.**

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{6}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}\right]$$ **Divide out the common h.**

$$= nx^{n-1}$$ **Take the limit. Only the first term remains.** ■

NOTE $\frac{d}{dx}x^n = nx^{n-1}$ is true not only for positive integers n but also for any real number n. But the proof requires future results. As these are developed, we will expand the Power Rule to include an ever-widening set of numbers until we arrive at the fact it is true when n is a real number.

EXAMPLE 2 Differentiating a Power Function

(a) $\frac{d}{dx}x^5 = 5x^4$ **(b)** If $g(x) = x^{10}$, then $g'(x) = 10x^9$. ■

NOW WORK Problem 1 and AP® Practice Problem 1.

But what if we want to find the derivative of the function $f(x) = ax^n$ when $a \neq 1$? The next theorem, called the *Constant Multiple Rule*, provides a way.

THEOREM Constant Multiple Rule

If a function f is differentiable and k is a constant, then $F(x)=kf(x)$ is a function that is differentiable and

$$\boxed{F'(x)=kf'(x)}$$

IN WORDS The derivative of a constant times a differentiable function f equals the constant times the derivative of f.

Proof Use the definition of a derivative, Form (2).

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{kf(x+h)-kf(x)}{h}$$
$$=\lim_{h\to 0}\frac{k[f(x+h)-f(x)]}{h}=k\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=k\cdot f'(x)$$ ■

Using Leibniz notation, the Constant Multiple Rule takes the form

$$\boxed{\frac{d}{dx}[kf(x)]=k\left[\frac{d}{dx}f(x)\right]}$$

A change in the symbol used for the independent variable does not affect the derivative formula. For example, $\frac{d}{dt}t^2=2t$ and $\frac{d}{du}u^5=5u^4$.

EXAMPLE 3 Differentiating a Constant Times a Power Function

Find the derivative of each function:

(a) $f(x)=5x^3$ **(b)** $g(u)=-\frac{1}{2}u^2$ **(c)** $u(x)=\pi^4x^3$

Solution

Notice that each of these functions involves the product of a constant and a power function. So, we use the Constant Multiple Rule followed by the Simple Power Rule.

(a) $f(x)=5\cdot x^3$, so $f'(x)=5\left[\frac{d}{dx}x^3\right]=5\cdot 3x^2=15x^2$

(b) $g(u)=-\frac{1}{2}\cdot u^2$, so $g'(u)=-\frac{1}{2}\cdot\frac{d}{du}u^2=-\frac{1}{2}\cdot 2u^1=-u$

(c) $u(x)=\pi^4x^3$, so $u'(x)\underset{\substack{\uparrow\\ \pi\text{ is a constant}}}{=}\pi^4\cdot\frac{d}{dx}x^3=\pi^4\cdot 3x^2=3\pi^4x^2$ ■

NOW WORK Problem 31.

3 Differentiate the Sum and the Difference of Two Functions

We can find the derivative of a function that is the sum of two functions whose derivatives are known by adding the derivatives of each function.

THEOREM Sum Rule

If two functions f and g are differentiable and if $F(x)=f(x)+g(x)$, then F is differentiable and

$$\boxed{F'(x)=f'(x)+g'(x)}$$

IN WORDS The derivative of the sum of two differentiable functions equals the sum of their derivatives. That is, $(f+g)'=f'+g'$.

Proof If $F(x) = f(x) + g(x)$, then

$$\begin{aligned} F(x+h) - F(x) &= [f(x+h) + g(x+h)] - [f(x) + g(x)] \\ &= [f(x+h) - f(x)] + [g(x+h) - g(x)] \end{aligned}$$

So, the derivative of F is

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \qquad \text{The limit of a sum is the sum of the limits.} \\ &= f'(x) + g'(x) \end{aligned}$$ ■

In Leibniz notation, the Sum Rule takes the form

$$\boxed{\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)}$$

EXAMPLE 4 Differentiating the Sum of Two Functions

Find the derivative of $f(x) = 3x^2 + 8$.

Solution

Here f is the sum of $3x^2$ and 8. So, we begin by using the Sum Rule.

$$f'(x) = \frac{d}{dx}(3x^2 + 8) \underset{\substack{\uparrow \\ \text{Sum Rule}}}{=} \frac{d}{dx}(3x^2) + \frac{d}{dx} 8 \underset{\substack{\uparrow \\ \text{Constant Multiple} \\ \text{Rule}}}{=} 3\frac{d}{dx} x^2 + 0 \underset{\substack{\uparrow \\ \text{Simple} \\ \text{Power Rule}}}{=} 3 \cdot 2x = 6x$$ ■

NOW WORK Problem 7 and AP® Practice Problem 6.

THEOREM Difference Rule

If the functions f and g are differentiable and if $F(x) = f(x) - g(x)$, then F is differentiable, and $F'(x) = f'(x) - g'(x)$. That is,

$$\boxed{\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)}$$

IN WORDS The derivative of the difference of two differentiable functions is the difference of their derivatives. That is, $(f - g)' = f' - g'$.

The proof of the Difference Rule is left as an exercise. (See Problem 78.)

The Sum and Difference Rules extend to sums (or differences) of more than two functions. That is, if the functions $f_1, f_2, \ldots, f_n$ are all differentiable, and $a_1, a_2, \ldots, a_n$ are constants, then

$$\boxed{\frac{d}{dx}[a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)] = a_1 \frac{d}{dx} f_1(x) + a_2 \frac{d}{dx} f_2(x) + \cdots + a_n \frac{d}{dx} f_n(x)}$$

Combining the rules for finding the derivative of a constant, a power function, and a sum or difference allows us to differentiate any polynomial function.

EXAMPLE 5 Differentiating a Polynomial Function

(a) Find the derivative of $f(x) = 2x^4 - 6x^2 + 2x - 3$.

(b) What is $f'(2)$?

(c) Find the slope of the tangent line to the graph of f at the point $(1, -5)$.

(d) Find an equation of the tangent line to the graph of f at the point $(1, -5)$.

(e) Find an equation of the normal line to the graph of f at the point $(1, -5)$.

(f) Use technology to graph f, the tangent line, and the normal line to the graph of f at the point $(1, -5)$ on the same screen.

Solution

(a) $$f'(x) = \frac{d}{dx}(2x^4 - 6x^2 + 2x - 3) \underset{\uparrow}{=} \frac{d}{dx}(2x^4) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}3$$

Sum/Difference Rules

$$\underset{\uparrow}{=} 2 \cdot \frac{d}{dx}x^4 - 6 \cdot \frac{d}{dx}x^2 + 2 \cdot \frac{d}{dx}x - 0$$

Constant Multiple Rule

$$\underset{\uparrow}{=} 2 \cdot 4x^3 - 6 \cdot 2x + 2 \cdot 1 \underset{\uparrow}{=} 8x^3 - 12x + 2$$

Simple Power Rule Simplify

(b) $f'(2) = 8 \cdot 2^3 - 12 \cdot 2 + 2 = 64 - 24 + 2 = 42.$

(c) The slope of the tangent line at the point $(1, -5)$ equals $f'(1)$.

$$f'(1) = 8 \cdot 1^3 - 12 \cdot 1 + 2 = 8 - 12 + 2 = -2$$

(d) Use the point-slope form of an equation of a line to find an equation of the tangent line at $(1, -5)$.

$$y - (-5) = -2(x - 1)$$
$$y = -2(x - 1) - 5 = -2x + 2 - 5 = -2x - 3$$

The line $y = -2x - 3$ is tangent to the graph of $f(x) = 2x^4 - 6x^2 + 2x - 3$ at the point $(1, -5)$.

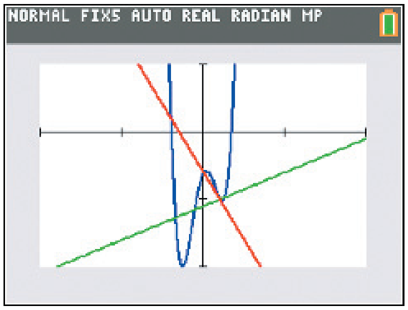
(e) Since the normal line and the tangent line at the point $(1, -5)$ on the graph of f are perpendicular and the slope of the tangent line is -2, the slope of the normal line is $\frac{1}{2}$.

Use the point-slope form of an equation of a line to find an equation of the normal line.

$$y - (-5) = \frac{1}{2}(x - 1)$$
$$y = \frac{1}{2}(x - 1) - 5 = \frac{1}{2}x - \frac{1}{2} - 5 = \frac{1}{2}x - \frac{11}{2}$$

The line $y = \frac{1}{2}x - \frac{11}{2}$ is normal to the graph of f at the point $(1, -5)$.

(f) The graphs of f, the tangent line, and the normal line to f at $(1, -5)$ are shown in Figure 26. ■



$[-10, 10] \times [-10, 5]$

Figure 26 $f(x) = 2x^4 - 6x^2 + 2x - 3$

NOW WORK Problem 33 and AP® Practice Problems 2, 5, 10, and 11.

In some applications, we need to solve equations or inequalities involving the derivative of a function.

EXAMPLE 6 Solving Equations and Inequalities Involving Derivatives

(a) Find the points on the graph of $f(x) = 4x^3 - 12x^2 + 2$, where f has a horizontal tangent line.

(b) Where is $f'(x) > 0$? Where is $f'(x) < 0$?

Solution

(a) The slope of a horizontal tangent line is 0. Since the derivative of f equals the slope of the tangent line, we need to find the numbers x for which $f'(x) = 0$.

$$f'(x) = 12x^2 - 24x = 12x(x - 2)$$

$$12x(x - 2) = 0 \qquad f'(x) = 0$$

$$x = 0 \text{ or } x = 2 \qquad \text{Solve.}$$

At the points $(0, f(0)) = (0, 2)$ and $(2, f(2)) = (2, -14)$, the graph of the function $f(x) = 4x^3 - 12x^2 + 2$ has horizontal tangent lines.

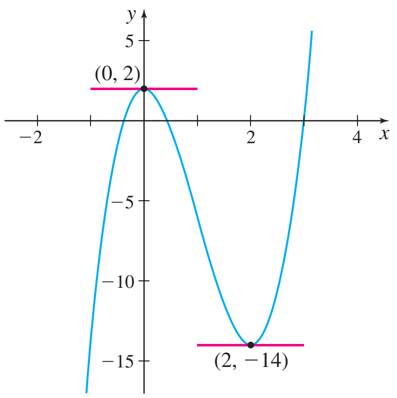


Figure 27 $f(x) = 4x^3 - 12x^2 + 2$

(b) Since $f'(x) = 12x(x-2)$ and we want to solve the inequalities $f'(x) > 0$ and $f'(x) < 0$, we use the zeros of f', 0 and 2, and form a table using the intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$. ■

TABLE 2

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f'(x) = 12x(x-2)$	Positive	Negative	Positive

We conclude $f'(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$ and $f'(x) < 0$ on $(0, 2)$.

Figure 27 shows the graph of f and the two horizontal tangent lines.

NOW WORK Problem 37 and AP® Practice Problem 3.

4 Differentiate the Exponential Function $y = e^x$

None of the differentiation rules developed so far allow us to find the derivative of an exponential function. To differentiate $f(x) = a^x$, we need to return to the definition of a derivative.

NEED TO REVIEW? Exponential functions are discussed in Section P.5, pp. 41–44.

We begin by making some general observations about the derivative of $f(x) = a^x$, $a > 0$ and $a \neq 1$. We then use these observations to find the derivative of the exponential function $y = e^x$.

Suppose $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The derivative of f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \underset{\uparrow}{=} \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$a^{x+h} = a^x \cdot a^h$

$$\underset{\uparrow}{=} \lim_{h \to 0} \left[a^x \cdot \frac{a^h - 1}{h} \right] = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}$$

Factor out a^x.

provided $\lim_{h \to 0} \frac{a^h - 1}{h}$ exists.

Three observations about the derivative of $f(x) = a^x$ are significant:

- $f'(0) = a^0 \lim_{h \to 0} \frac{a^h - 1}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}$.
- $f'(x)$ is a multiple of a^x. In fact, $\frac{d}{dx} a^x = f'(0) \cdot a^x$.
- If $f'(0)$ exists, then $f'(x)$ exists, and the domain of f' is the same as that of $f(x) = a^x$, all real numbers.

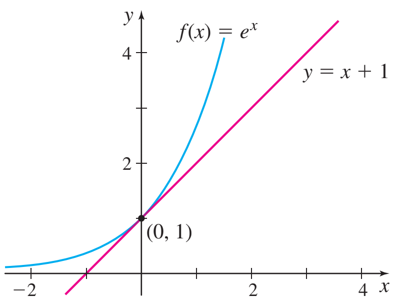


Figure 28

NEED TO REVIEW? The number e is discussed in Section P.5, pp. 44–45.

The slope of the tangent line to the graph of $f(x) = a^x$ at the point $(0, 1)$ is $f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$, and the value of this limit depends on the base a. In Section P.5, the number e was defined as that number for which the slope of the tangent line to the graph of $y = a^x$ at the point $(0, 1)$ equals 1. That is, if $f(x) = e^x$, then $f'(0) = 1$ so that

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

Figure 28 shows $f(x) = e^x$ and the tangent line $y = x + 1$ with slope 1 at the point $(0, 1)$.

Since $\frac{d}{dx}a^x = f'(0)\cdot a^x$, if $f(x)=e^x$, then $\frac{d}{dx}e^x = f'(0)\cdot e^x = 1\cdot e^x = e^x$.

THEOREM Derivative of the Exponential Function $y=e^x$

The derivative of the exponential function $y=e^x$ is

$$\boxed{y' = \frac{d}{dx}e^x = e^x} \qquad (1)$$

CALC CLIP

EXAMPLE 7 Differentiating an Expression Involving $y=e^x$

Find the derivative of $f(x)=4e^x+x^3$.

Solution

The function f is the sum of $4e^x$ and x^3. Then

$$f'(x) = \underset{\text{Sum Rule}}{\frac{d}{dx}(4e^x+x^3) = \frac{d}{dx}(4e^x)} + \underset{\substack{\text{Constant Multiple Rule;}\\ \text{Simple Power Rule}}}{\frac{d}{dx}x^3 = 4\frac{d}{dx}e^x} + \underset{\text{Use (1).}}{3x^2 = 4e^x} + 3x^2 \quad \blacksquare$$

NOTE We have not forgotten $y=\ln x$. Here is its derivative:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Use this result for now. We do not have the necessary mathematics to prove it until Chapter 3.

NOW WORK Problem 25 and AP® Practice Problems 4 and 9.

Now we know $\frac{d}{dx}e^x = e^x$. To find the derivative of $f(x)=a^x$, $a>0$ and $a\neq 1$, we need more information. See Chapter 3.

2.3 Assess Your Understanding

Concepts and Vocabulary

PAGE 185 **1.** $\frac{d}{dx}\pi^2 =$ ____; $\frac{d}{dx}x^3 =$ ____.

2. When n is a positive integer, the Simple Power Rule states that $\frac{d}{dx}x^n =$ ____.

3. *True or False* The derivative of a power function of degree greater than 1 is also a power function.

4. If k is a constant and f is a differentiable function, then $\frac{d}{dx}[kf(x)] =$ ____.

5. The derivative of $f(x)=e^x$ is ____.

6. *True or False* The derivative of an exponential function $f(x)=a^x$, where $a>0$ and $a\neq 1$, is always a constant multiple of a^x.

Skill Building

In Problems 7–26, find the derivative of each function using the formulas of this section. (a, b, c, and d, when they appear, are constants.)

PAGE 187 **7.** $f(x)=3x+\sqrt{2}$

8. $f(x)=5x-\pi$

9. $f(x)=x^2+3x+4$

10. $f(x)=4x^4+2x^2-2$

11. $f(u)=8u^5-5u+1$

12. $f(u)=9u^3-2u^2+4u+4$

13. $f(s)=as^3+\frac{3}{2}s^2$

14. $f(s)=4-\pi s^2$

15. $f(t)=\frac{1}{3}(t^5-8)$

16. $f(x)=\frac{1}{5}(x^7-3x^2+2)$

17. $f(t)=\frac{t^3+2}{5}$

18. $f(x)=\frac{x^7-5x}{9}$

19. $f(x)=\frac{x^3+2x+1}{7}$

20. $f(x)=\frac{1}{a}(ax^2+bx+c)$, $a\neq 0$

21. $f(x)=ax^2+bx+c$

22. $f(x)=ax^3+bx^2+cx+d$

23. $f(x)=4e^x$

24. $f(x)=-\frac{1}{2}e^x$

PAGE 190 **25.** $f(u)=5u^2-2e^u$

26. $f(u)=3e^u+10$

In Problems 27–32, find each derivative.

27. $\frac{d}{dt}\left(\sqrt{3}\,t+\frac{1}{2}\right)$

28. $\frac{d}{dt}\left(\frac{2t^4-5}{8}\right)$

29. $\frac{dA}{dR}$ if $A(R)=\pi R^2$

30. $\frac{dC}{dR}$ if $C=2\pi R$

PAGE 186 **31.** $\frac{dV}{dr}$ if $V=\frac{4}{3}\pi r^3$

32. $\frac{dP}{dT}$ if $P=0.2T$

In Problems 33–36:

(a) Find the slope of the tangent line to the graph of each function f at the indicated point.

(b) Find an equation of the tangent line at the point.

(c) Find an equation of the normal line at the point.

(d) Graph f and the tangent line and normal line found in (b) and (c) on the same set of axes.

PAGE 188 **33.** $f(x)=x^3+3x-1$ at $(0,-1)$

34. $f(x)=x^4+2x-1$ at $(1,2)$

35. $f(x)=e^x+5x$ at $(0,1)$

36. $f(x)=4-e^x$ at $(0,3)$

In Problems 37–42:

(a) *Find the points, if any, at which the graph of each function f has a horizontal tangent line.*

(b) *Find an equation for each horizontal tangent line.*

(c) *Solve the inequality $f'(x) > 0$.*

(d) *Solve the inequality $f'(x) < 0$.*

(e) *Graph f and any horizontal lines found in (b) on the same set of axes.*

(f) *Describe the graph of f for the results obtained in parts (c) and (d).*

PAGE 189

37. $f(x) = 3x^2 - 12x + 4$ **38.** $f(x) = x^2 + 4x - 3$

39. $f(x) = x + e^x$ **40.** $f(x) = 2e^x - 1$

41. $f(x) = x^3 - 3x + 2$ **42.** $f(x) = x^4 - 4x^3$

43. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^3 - t + 1$. Find the velocity of the object at $t = 0$ and at $t = 5$.

44. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^4 - t^3 + 1$. Find the velocity of the object at $t = 0$ and at $t = 1$.

Rectilinear Motion *In Problems 45 and 46, each position function gives the signed distance s from the origin at time t of an object in rectilinear motion:*

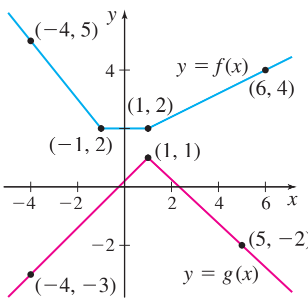
(a) *Find the velocity v of the object at any time t.*

(b) *When is the velocity of the object 0?*

45. $s(t) = 2 - 5t + t^2$ **46.** $s(t) = t^3 - \frac{9}{2}t^2 + 6t + 4$

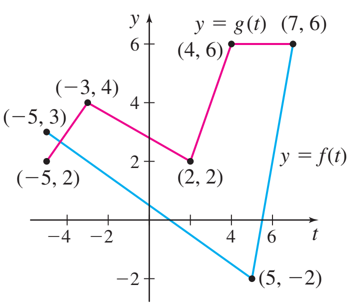
In Problems 47 and 48, use the graphs to find each derivative.

47. Let $u(x) = f(x) + g(x)$ and $v(x) = f(x) - g(x)$.



(a) $u'(0)$ **(b)** $u'(4)$
(c) $v'(-2)$ **(d)** $v'(6)$
(e) $3u'(5)$ **(f)** $-2v'(3)$

48. Let $F(t) = f(t) + g(t)$ and $G(t) = g(t) - f(t)$.



(a) $F'(0)$ **(b)** $F'(3)$
(c) $F'(-4)$ **(d)** $G'(-2)$
(e) $G'(-1)$ **(f)** $G'(6)$

In Problems 49 and 50, for each function f:

(a) *Find $f'(x)$ by expanding $f(x)$ and differentiating the polynomial.*

CAS **(b)** *Find $f'(x)$ using a CAS.*

(c) *Show that the results found in parts (a) and (b) are equivalent.*

49. $f(x) = (2x - 1)^3$ **50.** $f(x) = (x^2 + x)^4$

Applications and Extensions

In Problems 51–56, find each limit.

51. $\lim_{h \to 0} \frac{5\left(\frac{1}{2} + h\right)^8 - 5\left(\frac{1}{2}\right)^8}{h}$ **52.** $\lim_{h \to 0} \frac{6(2+h)^5 - 6 \cdot 2^5}{h}$

53. $\lim_{h \to 0} \frac{\sqrt{3}(8+h)^5 - \sqrt{3} \cdot 8^5}{h}$ **54.** $\lim_{h \to 0} \frac{\pi(1+h)^{10} - \pi}{h}$

55. $\lim_{h \to 0} \frac{a(x+h)^3 - ax^3}{h}$ **56.** $\lim_{h \to 0} \frac{b(x+h)^n - bx^n}{h}$

In Problems 57–62, find an equation of the tangent line(s) to the graph of the function f that is (are) parallel to the line L.

57. $f(x) = 3x^2 - x$; L: $y = 5x$

58. $f(x) = 2x^3 + 1$; L: $y = 6x - 1$

59. $f(x) = e^x$; L: $y - x - 5 = 0$

60. $f(x) = -2e^x$; L: $y + 2x - 8 = 0$

61. $f(x) = \frac{1}{3}x^3 - x^2$; L: $y = 3x - 2$

62. $f(x) = x^3 - x$; L: $x + y = 0$

63. Tangent Lines Let $f(x) = 4x^3 - 3x - 1$.

(a) Find an equation of the tangent line to the graph of f at $x = 2$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = x + 12$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = x + 12$, and any tangent lines found in (c) on the same screen.

64. Tangent Lines Let $f(x) = x^3 + 2x^2 + x - 1$.

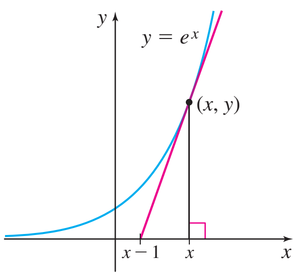
(a) Find an equation of the tangent line to the graph of f at $x = 0$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = 3x - 2$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = 3x - 2$, and any tangent lines found in (c) on the same screen.

65. Tangent Line Show that the line perpendicular to the x-axis and containing the point (x, y) on the graph of $y = e^x$ and the tangent line to the graph of $y = e^x$ at the point (x, y) intersect the x-axis 1 unit apart. See the figure.



66. Tangent Line Show that the tangent line to the graph of $y = x^n$, $n \geq 2$ an integer, at $(1, 1)$ has y-intercept $1 - n$.

67. Tangent Lines If n is an odd positive integer, show that the tangent lines to the graph of $y = x^n$ at $(1, 1)$ and at $(-1, -1)$ are parallel.

68. Tangent Line If the line $3x - 4y = 0$ is tangent to the graph of $y = x^3 + k$ in the first quadrant, find k.

69. Tangent Line Find the constants a, b, and c so that the graph of $y = ax^2 + bx + c$ contains the point $(-1, 1)$ and is tangent to the line $y = 2x$ at $(0, 0)$.

70. Tangent Line Let T be the tangent line to the graph of $y = x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$. At what other point Q on the graph of $y = x^3$ does the line T intersect the graph? What is the slope of the tangent line at Q?

71. Military Tactics A dive bomber is flying from right to left along the graph of $y = x^2$. When a rocket bomb is released, it follows a path that is approximately along the tangent line. Where should the pilot release the bomb if the target is at $(1, 0)$?

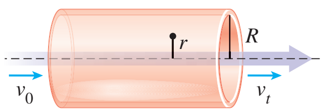
72. Military Tactics Answer the question in Problem 71 if the plane is flying from right to left along the graph of $y = x^3$.

73. Fluid Dynamics The velocity v of a liquid flowing through a cylindrical tube is given by the **Hagen–Poiseuille equation** $v = k(R^2 - r^2)$, where R is the radius of the tube, k is a constant that depends on the length of the tube and the velocity of the liquid at its ends, and r is the variable distance of the liquid from the center of the tube. See the figure below.

(a) Find the rate of change of v with respect to r at the center of the tube.

(b) What is the rate of change halfway from the center to the wall of the tube?

(c) What is the rate of change at the wall of the tube?



74. Rate of Change Water is leaking out of a swimming pool that measures 20 ft by 40 ft by 6 ft. The amount of water in the pool at a time t is $W(t) = 35{,}000 - 20t^2$ gallons, where t equals the number of hours since the pool was last filled. At what rate is the water leaking when $t = 2$ h?

75. Luminosity of the Sun The luminosity L of a star is the rate at which it radiates energy. This rate depends on the temperature T and surface area A of the star's photosphere (the gaseous surface that emits the light). Luminosity is modeled by the equation $L = \sigma A T^4$, where σ is a constant known as the **Stefan–Boltzmann constant**, and T is expressed in the absolute (Kelvin) scale for which 0 K is absolute zero. As with most stars, the Sun's temperature has gradually increased over the 6 billion years of its existence, causing its luminosity to slowly increase.

(a) Find the rate at which the Sun's luminosity changes with respect to the temperature of its photosphere. Assume that the surface area A remains constant.

(b) Find the rate of change at the present time. The temperature of the photosphere is currently 5800 K (10,000 °F), the radius of the photosphere is $r = 6.96 \times 10^8$ m, and $\sigma = 5.67 \times 10^{-8} \dfrac{\text{W}}{\text{m}^2\,\text{K}^4}$.

(c) Assuming that the rate found in (b) remains constant, how much would the luminosity change if its photosphere temperature increased by 1 K (1 °C or 1.8 °F)? Compare this change to the present luminosity of the Sun.

76. Medicine: Poiseuille's Equation The French physician Poiseuille discovered that the volume V of blood (in cubic centimeters per unit time) flowing through an artery with inner radius R (in centimeters) can be modeled by

$$V(R) = kR^4$$

where $k = \dfrac{\pi}{8\nu l}$ is constant (here ν represents the viscosity of blood and l is the length of the artery).

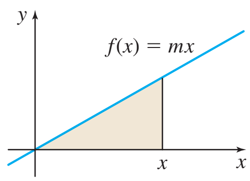
(a) Find the rate of change of the volume V of blood flowing through the artery with respect to the radius R.

(b) Find the rate of change when $R = 0.03$ and when $R = 0.04$.

(c) If the radius of a partially clogged artery is increased from 0.03 to 0.04 cm, estimate the effect on the rate of change of the volume V with respect to R of the blood flowing through the enlarged artery.

(d) How do you interpret the results found in (b) and (c)?

77. Derivative of an Area Let $f(x) = mx$, $m > 0$. Let $F(x)$, $x > 0$, be defined as the area of the shaded region in the figure. Find $F'(x)$.



78. The Difference Rule Prove that if f and g are differentiable functions and if $F(x) = f(x) - g(x)$, then

$$F'(x) = f'(x) - g'(x)$$

79. Simple Power Rule Let $f(x) = x^n$, where n is a positive integer. Use a factoring principle to show that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = nc^{n-1}$$

80. Normal Lines For what nonnegative number b is the line given by $y = -\frac{1}{3}x + b$ normal to the graph of $y = x^3$?

81. Normal Lines Let N be the normal line to the graph of $y = x^2$ at the point $(-2, 4)$. At what other point Q does N meet the graph?

Challenge Problems

82. Tangent Line Find a, b, c, d so that the tangent line to the graph of the cubic $y = ax^3 + bx^2 + cx + d$ at the point $(1, 0)$ is $y = 3x - 3$ and at the point $(2, 9)$ is $y = 18x - 27$.

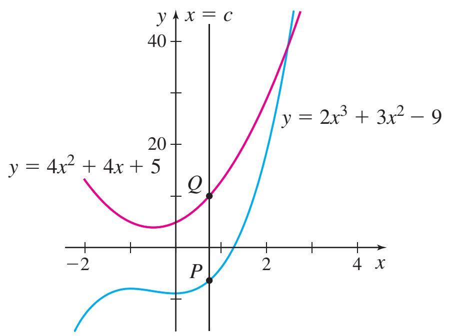
83. Tangent Line Find the fourth degree polynomial that contains the origin and to which the line $x + 2y = 14$ is tangent at both $x = 4$ and $x = -2$.

84. Tangent Lines Find equations for all the lines containing the point $(1, 4)$ that are tangent to the graph of $y = x^3 - 10x^2 + 6x - 2$. At what points do each of the tangent lines touch the graph?

85. The line $x = c$, where $c > 0$, intersects the cubic $y = 2x^3 + 3x^2 - 9$ at the point P and intersects the parabola $y = 4x^2 + 4x + 5$ at the point Q, as shown in the figure on the right.

(a) If the line tangent to the cubic at the point P is parallel to the line tangent to the parabola at the point Q, find the number c.

(b) Write an equation for each of the two tangent lines described in (a).



86. $f(x) = Ax^2 + B$, $A > 0$.

(a) Find c, $c > 0$, in terms of A so that the tangent lines to the graph of f at $(c, f(c))$ and $(-c, f(-c))$ are perpendicular.

(b) Find the slopes of the tangent lines in (a).

(c) Find the coordinates, in terms of A and B, of the point of intersection of the tangent lines in (a).

Preparing for the AP® Exam

AP® Practice Problems

PAGE 185 **1.** If $g(x) = x$, then $g'(7) =$

(A) 0 (B) 1 (C) 7 (D) $\dfrac{49}{2}$

PAGE 188 **2.** The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

(A) -1 (B) 2 (C) -2 (D) -4

PAGE 189 **3.** An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

(A) -3 (B) 3 (C) $\dfrac{3}{2}$ (D) 7

PAGE 190 **4.** If $f(x) = e^x$, then $\ln(f'(3)) =$

(A) 3 (B) 0 (C) e^3 (D) $\ln 3$

PAGE 188 **5.** An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

(A) $x + 2y = 12$ (B) $x - 2y = 8$
(C) $2x + y = -9$ (D) $x + 2y = 8$

PAGE 187 **6.** The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

(A) $\dfrac{3}{32}$ (B) $\dfrac{3}{16}$ (C) $\dfrac{3}{64}$ (D) $\dfrac{9}{64}$

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 (D) $f'(4)$ does not exist.

PAGE 187 **8.** The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$.
What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 (C) 604.02 (D) 1020

PAGE 190 **9.** $\dfrac{d}{dx}(5 \ln x) =$

(A) $\dfrac{1}{5x}$ (B) $5e^x$ (C) $-\dfrac{5}{\ln x}$ (D) $\dfrac{5}{x}$

PAGE 188 **10.** For the function $f(x) = x^2 + 4$

(a) Find $f'(1)$.

(b) Find an equation of the tangent line to the graph of f at $x = 1$.

(c) Find $f'(-4)$.

(d) Find an equation of the tangent line to the graph of f at $x = -4$.

(e) Find the point of intersection of the two tangent lines found in (b) and (d).

PAGE 188 **11.** Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$
(C) $y = 2x + 1.678$ (D) $y = 2x - 2.929$

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Differentiate the product of two functions** **(p. 194)**
2. **Differentiate the quotient of two functions** **(p. 196)**
3. **Find higher-order derivatives** **(p. 198)**
4. **Find the acceleration of an object in rectilinear motion** **(p. 200)**

In this section, we obtain formulas for differentiating products and quotients of functions. As it turns out, the formulas are not what we might expect. The derivative of the product of two functions is *not* the product of their derivatives, and the derivative of the quotient of two functions is *not* the quotient of their derivatives.

1 Differentiate the Product of Two Functions

Consider the two functions $f(x)=2x$ and $g(x)=x^3$. Both are differentiable, and their derivatives are $f'(x)=2$ and $g'(x)=3x^2$. Form the product

$$F(x)=f(x)g(x)=2x\cdot x^3=2x^4$$

Now find F' using the Constant Multiple Rule and the Simple Power Rule.

$$F'(x)=2\cdot 4x^3=8x^3$$

Notice that $f'(x)g'(x)=2\cdot 3x^2=6x^2$ is not equal to $F'(x)=\dfrac{d}{dx}[f(x)g(x)]=8x^3$. We conclude that the derivative of a product of two functions is *not* the product of their derivatives.

To find the derivative of the product of two differentiable functions f and g, we let $F(x)=f(x)g(x)$ and use the definition of a derivative, namely,

$$F'(x)=\lim_{h\to 0}\frac{[f(x+h)g(x+h)]-[f(x)g(x)]}{h}$$

We can express F' in an equivalent form that contains the difference quotients for f and g, by subtracting and adding $f(x+h)g(x)$ to the numerator.

$$F'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+[f(x+h)-f(x)]g(x)}{h}$$ **Group and factor.**

$$=\left[\lim_{h\to 0}f(x+h)\right]\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]+\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0}g(x)\right]$$ **Use properties of limits.**

$$=\left[\lim_{h\to 0}f(x+h)\right]g'(x)+f'(x)\left[\lim_{h\to 0}g(x)\right]$$ **Definition of a derivative.**

$$=f(x)g'(x)+f'(x)g(x)$$ **$\lim_{h\to 0} g(x)=g(x)$ since h is not present. Since f is differentiable, it is continuous, so $\lim_{h\to 0} f(x+h)=f(x)$.**

We have proved the following theorem.

THEOREM Product Rule

If f and g are differentiable functions and if $F(x) = f(x)g(x)$, then F is differentiable, and the derivative of the product F is

$$\boxed{F'(x) = [f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)}$$

In Leibniz notation, the Product Rule has the form

$$\boxed{\frac{d}{dx}F(x) = \frac{d}{dx}[f(x)g(x)] = f(x)\left[\frac{d}{dx}g(x)\right] + \left[\frac{d}{dx}f(x)\right]g(x)}$$

IN WORDS The derivative of the product of two differentiable functions equals the first function times the derivative of the second function plus the derivative of the first function times the second function. That is,

$$(fg)' = f(g') + (f')g$$

EXAMPLE 1 Differentiating the Product of Two Functions

Find y' if $y = (1 + x^2)e^x$.

Solution

The function y is the product of two functions: a polynomial, $f(x) = 1 + x^2$, and the exponential function, $g(x) = e^x$. By the Product Rule,

$$y' = \frac{d}{dx}[(1 + x^2)e^x] \underset{\substack{\uparrow \\ \text{Product Rule}}}{=} (1 + x^2)\left[\frac{d}{dx}e^x\right] + \left[\frac{d}{dx}(1 + x^2)\right]e^x = (1 + x^2)e^x + 2xe^x$$

At this point, we have found the derivative, but it is customary to simplify the answer. Then

$$y' \underset{\substack{\uparrow \\ \text{Factor out } e^x.}}{=} (1 + x^2 + 2x)e^x \underset{\substack{\uparrow \\ \text{Factor.}}}{=} (x + 1)^2 e^x$$ ■

NOW WORK Problem 9 and AP® Practice Problem 4.

Do not use the Product Rule unnecessarily! When one of the factors is a constant, use the Constant Multiple Rule. For example, it is easier to work

$$\frac{d}{dx}[5(x^2 + 1)] = 5\frac{d}{dx}(x^2 + 1) = 5 \cdot 2x = 10x$$

than it is to work

$$\frac{d}{dx}[5(x^2 + 1)] = 5\frac{d}{dx}(x^2 + 1) + \left[\frac{d}{dx}5\right](x^2 + 1) = 5 \cdot 2x + 0 \cdot (x^2 + 1) = 10x$$

Also, it is easier to simplify $f(x) = x^2(4x - 3)$ before finding the derivative. That is, it is easier to work

$$\frac{d}{dx}[x^2(4x - 3)] = \frac{d}{dx}(4x^3 - 3x^2) = 12x^2 - 6x$$

than it is to use the Product Rule

$$\frac{d}{dx}[x^2(4x - 3)] = x^2\frac{d}{dx}(4x - 3) + \left(\frac{d}{dx}x^2\right)(4x - 3) = (x^2)(4) + (2x)(4x - 3)$$
$$= 4x^2 + 8x^2 - 6x = 12x^2 - 6x$$

CALC CLIP

EXAMPLE 2 Differentiating a Product in Two Ways

Find the derivative of $F(v) = (5v^2 - v + 1)(v^3 - 1)$ in two ways:

(a) By using the Product Rule

(b) By multiplying the factors of the function before finding its derivative.

Solution

(a) F is the product of the two functions $f(v) = 5v^2 - v + 1$ and $g(v) = v^3 - 1$. Using the Product Rule, we get

$$\begin{aligned} F'(v) &= (5v^2 - v + 1)\left[\frac{d}{dv}(v^3 - 1)\right] + \left[\frac{d}{dv}(5v^2 - v + 1)\right](v^3 - 1) \\ &= (5v^2 - v + 1)(3v^2) + (10v - 1)(v^3 - 1) \\ &= 15v^4 - 3v^3 + 3v^2 + 10v^4 - 10v - v^3 + 1 \\ &= 25v^4 - 4v^3 + 3v^2 - 10v + 1 \end{aligned}$$

(b) Here we multiply the factors of F before differentiating.

$$F(v) = (5v^2 - v + 1)(v^3 - 1) = 5v^5 - v^4 + v^3 - 5v^2 + v - 1$$

Then

$$F'(v) = 25v^4 - 4v^3 + 3v^2 - 10v + 1$$ ■

Notice that the derivative is the same whether you differentiate and then simplify, or whether you multiply the factors and then differentiate. Use the approach that you find easier.

NOW WORK Problem 13.

2 Differentiate the Quotient of Two Functions

The derivative of the quotient of two functions is *not* equal to the quotient of their derivatives. Instead, the derivative of the quotient of two functions is found using the *Quotient Rule*.

THEOREM Quotient Rule

If two functions f and g are differentiable and if $F(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then F is differentiable, and the derivative of the quotient F is

$$\boxed{F'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

In Leibniz notation, the Quotient Rule has the form

$$\boxed{\frac{d}{dx}F(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}}$$

IN WORDS The derivative of a quotient of two functions is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the denominator squared. That is,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Proof We use the definition of a derivative to find $F'(x)$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \underset{\uparrow \atop F(x) = \frac{f(x)}{g(x)}}{=} \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h[g(x+h)g(x)]}$$

We write F' in an equivalent form that contains the difference quotients for f and g by subtracting and adding $f(x)g(x)$ to the numerator.

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h[g(x+h)g(x)]}$$

Now group and factor the numerator.

$$F'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x) - f(x)[g(x+h) - g(x)]}{h[g(x+h)g(x)]}$$

$$= \lim_{h \to 0} \frac{\left[\dfrac{f(x+h) - f(x)}{h}\right] g(x) - f(x) \left[\dfrac{g(x+h) - g(x)}{h}\right]}{g(x+h)g(x)}$$

$$= \frac{\lim\limits_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h}\right] \cdot \lim\limits_{h \to 0} g(x) - \lim\limits_{h \to 0} f(x) \cdot \lim\limits_{h \to 0} \left[\dfrac{g(x+h) - g(x)}{h}\right]}{\lim\limits_{h \to 0} g(x+h) \cdot \lim\limits_{h \to 0} g(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ ■

RECALL Since g is differentiable, it is continuous; so, $\lim\limits_{h \to 0} g(x+h) = g(x)$.

EXAMPLE 3 Differentiating the Quotient of Two Functions

Find y' if $y = \dfrac{x^2+1}{2x-3}$.

Solution

The function y is the quotient of $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. Using the Quotient Rule, we have

$$y' = \frac{d}{dx} \frac{x^2+1}{2x-3} = \frac{\left[\dfrac{d}{dx}(x^2+1)\right](2x-3) - (x^2+1)\left[\dfrac{d}{dx}(2x-3)\right]}{(2x-3)^2}$$

$$= \frac{(2x)(2x-3) - (x^2+1)(2)}{(2x-3)^2} = \frac{4x^2 - 6x - 2x^2 - 2}{(2x-3)^2} = \frac{2x^2 - 6x - 2}{(2x-3)^2}$$

provided $x \neq \dfrac{3}{2}$. ■

NOW WORK Problem 23 and AP® Practice Problems 1, 2, 3, 7 and 8.

IN WORDS The derivative of the reciprocal of a function is the negative of the derivative of the denominator divided by the square of the denominator. That is,

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}.$$

COROLLARY Derivative of the Reciprocal of a Function

If a function g is differentiable, then

$$\boxed{\frac{d}{dx} \frac{1}{g(x)} = -\frac{\dfrac{d}{dx} g(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}} \qquad (1)$$

provided $g(x) \neq 0$.

The proof of the corollary is left as an exercise. (See Problem 98.)

EXAMPLE 4 Differentiating the Reciprocal of a Function

(a) $$\frac{d}{dx} \frac{1}{x^2+x} \underset{\substack{\uparrow \\ \text{Use (1).}}}{=} -\frac{\dfrac{d}{dx}(x^2+x)}{(x^2+x)^2} = -\frac{2x+1}{(x^2+x)^2}$$

(b) $$\frac{d}{dx} e^{-x} = \frac{d}{dx} \frac{1}{e^x} \underset{\substack{\uparrow \\ \text{Use (1).}}}{=} -\frac{\dfrac{d}{dx} e^x}{(e^x)^2} = -\frac{e^x}{e^{2x}} = -\frac{1}{e^x} = -e^{-x}$$ ■

NOW WORK Problem 25.

Notice that the derivative of the reciprocal of a function f is *not* the reciprocal of the derivative. That is,

$$\frac{d}{dx}\frac{1}{f(x)} \neq \frac{1}{f'(x)}$$

The rule for the derivative of the reciprocal of a function allows us to extend the Simple Power Rule to all integers. Here is the proof.

Suppose n is a negative integer and $x \neq 0$. Then $m = -n$ is a positive integer, and

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^m} \underset{\text{Use (1).}}{=} -\frac{\frac{d}{dx}x^m}{(x^m)^2} \underset{\text{Simple Power Rule}}{=} -\frac{mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{-m-1} \underset{\text{Substitute } n=-m.}{=} nx^{n-1} \quad \blacksquare$$

NOTE In Section 2.3 we proved the Simple Power Rule, $\frac{d}{dx}x^n = nx^{n-1}$ where n is a positive integer. Here we have extended the Simple Power Rule from positive integers to all integers. In Chapter 3 we extend the result to include all real numbers.

THEOREM Power Rule

The derivative of $y = x^n$, where n is any integer, is

$$\boxed{y' = \frac{d}{dx}x^n = nx^{n-1}}$$

EXAMPLE 5 Differentiating Using the Power Rule

(a) $\frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$

(b) $\frac{d}{du}\frac{1}{u^2} = \frac{d}{du}u^{-2} = -2u^{-3} = -\frac{2}{u^3}$

(c) $\frac{d}{ds}\frac{4}{s^5} = 4\frac{d}{ds}s^{-5} = 4\cdot(-5)\,s^{-6} = -20s^{-6} = -\frac{20}{s^6}$ ■

NOW WORK Problem 31 and AP® Practice Problem 5.

EXAMPLE 6 Using the Power Rule in Electrical Engineering

Ohm's Law states that the current I running through a wire is inversely proportional to the resistance R in the wire and can be written as $I = \frac{V}{R}$, where V is the voltage. Find the rate of change of I with respect to R when $V = 12$ volts.

Solution

The rate of change of I with respect to R is the derivative $\frac{dI}{dR}$. We write Ohm's Law with $V = 12$ as $I = \frac{V}{R} = 12R^{-1}$ and use the Power Rule.

$$\frac{dI}{dR} = \frac{d}{dR}(12R^{-1}) = 12\cdot\frac{d}{dR}R^{-1} = 12(-1R^{-2}) = -\frac{12}{R^2}$$

The minus sign in $\frac{dI}{dR}$ indicates that the current I decreases as the resistance R in the wire increases. ■

NOW WORK Problem 91.

3 Find Higher-Order Derivatives

Since the derivative f' is a function, it makes sense to ask about the derivative of f'. The derivative (if there is one) of f' is also a function called the **second derivative** of f and denoted by f'', read "f double prime."

By continuing in this fashion, we can find the **third derivative** of f, the **fourth derivative** of f, and so on, provided that these derivatives exist. Collectively, these are called **higher-order derivatives**.

Leibniz notation also can be used for higher-order derivatives. Table 3 summarizes the notation for higher-order derivatives.

TABLE 3

	Prime Notation		Leibniz Notation	
First Derivative	y'	$f'(x)$	$\dfrac{dy}{dx}$	$\dfrac{d}{dx}f(x)$
Second Derivative	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2}f(x)$
Third Derivative	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3}f(x)$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4}{dx^4}f(x)$
$\vdots$				
***n*th Derivative**	$y^{(n)}$	$f^{(n)}(x)$	$\dfrac{d^ny}{dx^n}$	$\dfrac{d^n}{dx^n}f(x)$

EXAMPLE 7 Finding Higher-Order Derivatives of a Power Function

Find the second, third, and fourth derivatives of $y = 2x^3$.

Solution

Use the Power Rule and the Constant Multiple Rule to find each derivative. The first derivative is

$$y' = \frac{d}{dx}(2x^3) = 2 \cdot \frac{d}{dx}x^3 = 2 \cdot 3x^2 = 6x^2$$

The next three derivatives are

$$\begin{aligned} y'' &= \frac{d^2}{dx^2}(2x^3) = \frac{d}{dx}(6x^2) = 6 \cdot \frac{d}{dx}x^2 = 6 \cdot 2x = 12x \\ y''' &= \frac{d^3}{dx^3}(2x^3) = \frac{d}{dx}(12x) = 12 \\ y^{(4)} &= \frac{d^4}{dx^4}(2x^3) = \frac{d}{dx}12 = 0 \end{aligned}$$

■

All derivatives of this function f of order 4 or more equal 0. This result can be generalized. For a power function f of degree n, where n is a positive integer,

$$\begin{aligned} f(x) &= x^n \\ f'(x) &= nx^{n-1} \\ f''(x) &= n(n-1)x^{n-2} \\ &\vdots \\ f^{(n)}(x) &= n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

NOTE If $n > 1$ is an integer, the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ is often written $n!$ and is read, "n factorial." The **factorial symbol** ! means $0! = 1$, $1! = 1$, and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$, where $n > 1$.

The nth-order derivative of $f(x) = x^n$ is a constant, so all derivatives of order greater than n equal 0.

It follows from this discussion that the nth derivative of a polynomial of degree n is a constant and that all derivatives of order $n + 1$ and higher equal 0.

NOW WORK Problem 41.

EXAMPLE 8 Finding Higher-Order Derivatives

Find the second and third derivatives of $y = (1 + x^2)e^x$.

Solution

In Example 1, we found that $y' = (1 + x^2)e^x + 2xe^x = (x^2 + 2x + 1)e^x$. To find y'', use the Product Rule with y'.

$$y'' = \frac{d}{dx}[(x^2+2x+1)e^x] \underset{\uparrow \text{ Product Rule}}{=} (x^2+2x+1)\left(\frac{d}{dx}e^x\right) + \left[\frac{d}{dx}(x^2+2x+1)\right]e^x$$

$$= (x^2+2x+1)e^x + (2x+2)e^x = (x^2+4x+3)e^x$$

$$y''' = \frac{d}{dx}[(x^2+4x+3)e^x] \underset{\uparrow \text{ Product Rule}}{=} (x^2+4x+3)\frac{d}{dx}e^x + \left[\frac{d}{dx}(x^2+4x+3)\right]e^x$$

$$= (x^2+4x+3)e^x + (2x+4)e^x = (x^2+6x+7)e^x$$ ■

NOW WORK Problem 45 and AP® Practice Problem 9.

4 Find the Acceleration of an Object in Rectilinear Motion

For an object in rectilinear motion whose signed distance s from the origin at time t is the position function $s = s(t)$, the derivative $s'(t)$ has a physical interpretation as the velocity of the object. The second derivative s'', which is the rate of change of the velocity, is called *acceleration*.

DEFINITION Acceleration

For an object in rectilinear motion, its signed distance s from the origin at time t is given by a position function $s = s(t)$. The first derivative $\frac{ds}{dt}$ is the velocity $v = v(t)$ of the object at time t.

The **acceleration** $a = a(t)$ of an object at time t is defined as the rate of change of velocity with respect to time. That is,

$$\boxed{a = a(t) = \frac{dv}{dt} = \frac{d}{dt}v = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}}$$

IN WORDS Acceleration is the second derivative of a position function with respect to time.

EXAMPLE 9 Analyzing Vertical Motion

A ball is propelled vertically upward from the ground with an initial velocity of 29.4 m/s. The height s (in meters) of the ball above the ground is approximately $s = s(t) = -4.9t^2 + 29.4t$, where t is the number of seconds that elapse from the moment the ball is released.

(a) What is the velocity of the ball at time t? What is its velocity at $t = 1$ s?

(b) When will the ball reach its maximum height?

(c) What is the maximum height the ball reaches?

(d) What is the acceleration of the ball at any time t?

(e) How long is the ball in the air?

(f) What is the velocity of the ball upon impact with the ground? What is its speed?

(g) What is the total distance traveled by the ball?

Solution

(a) Since $s = s(t) = -4.9t^2 + 29.4t$, then

$$v = v(t) = \frac{ds}{dt} = -9.8t + 29.4$$

$$v(1) = -9.8 + 29.4 = 19.6$$

At $t = 1$ s, the velocity of the ball is 19.6 m/s.

(b) The ball reaches its maximum height when $v(t) = 0$.

$$\begin{aligned} v(t) &= -9.8t + 29.4 = 0 \\ 9.8t &= 29.4 \\ t &= 3 \end{aligned}$$

The ball reaches its maximum height after 3 s.

(c) The maximum height is

$$s = s(3) = -4.9 \cdot 3^2 + 29.4 \cdot 3 = 44.1$$

The maximum height of the ball is 44.1 m.

(d) The acceleration of the ball at any time t is

$$a = a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-9.8t + 29.4) = -9.8 \text{ m/s}^2$$

NOTE *Speed* and *velocity* are not the same. Speed measures how fast an object is moving and is defined as the absolute value of its velocity. Velocity measures both the speed and the direction of an object and may be a positive number or a negative number or zero.

(e) There are two ways to answer the question "How long is the ball in the air?" *First way:* Since it takes 3 s for the ball to reach its maximum height, it follows that it will take another 3 s to reach the ground, for a total time of 6 s in the air. *Second way:* When the ball reaches the ground, $s = s(t) = 0$. Solve for t:

$$\begin{aligned} s(t) &= -4.9t^2 + 29.4t = 0 \\ t(-4.9t + 29.4) &= 0 \\ t = 0 \qquad \text{or} \qquad t &= \frac{29.4}{4.9} = 6 \end{aligned}$$

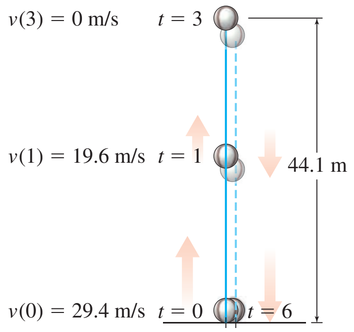


Figure 29

The ball is at ground level at $t = 0$ and at $t = 6$, so the ball is in the air for 6 s.

(f) Upon impact with the ground, $t = 6$ s. So the velocity is

$$v(6) = (-9.8)(6) + 29.4 = -29.4$$

Upon impact the direction of the ball is downward, and its speed is 29.4 m/s.

(g) The total distance traveled by the ball is

$$s(3) + s(3) = 2\,s(3) = 2(44.1) = 88.2 \text{ m}$$ ■

See Figure 29 for an illustration.

NOW WORK Problem 83 and AP® Practice Problem 6.

NOTE The Earth is not perfectly round; it bulges slightly at the equator, and its mass is not distributed uniformly. As a result, the acceleration of a freely falling body varies slightly.

In Example 9, the acceleration of the ball is constant. In fact, acceleration is the same for all falling objects at the same location, provided air resistance is not taken into account. In the sixteenth century, Galileo (1564–1642) discovered this by experimentation.* He also found that all falling bodies obey the law, stating that the distance s they fall when dropped is proportional to the square of the time t it takes to fall that distance, and that the constant of proportionality c is the same for all objects. That is,

$$s = -ct^2$$

*In a famous legend, Galileo dropped a feather and a rock from the top of the Leaning Tower of Pisa, to show that the acceleration due to gravity is constant. He expected them to fall together, but he failed to account for air resistance that slowed the feather. In July 1971, *Apollo* 15 astronaut David Scott repeated the experiment on the Moon, where there is no air resistance. He dropped a hammer and a falcon feather from his shoulder height. Both hit the Moon's surface at the same time. A video of this experiment may be found at the NASA website.

The velocity v of the falling object is

$$v = \frac{ds}{dt} = \frac{d}{dt}(-ct^2) = -2ct$$

and its acceleration a is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2c$$

which is a constant. Usually, we denote the constant $2c$ by g so $c = \frac{1}{2}g$. Then

$$\boxed{a = -g \qquad v = -gt \qquad s = -\frac{1}{2}gt^2}$$

The number g is called the **acceleration due to gravity**. For our planet, g is approximately 32 ft/s^2, or 9.8 m/s^2. On the planet Jupiter, $g \approx 26.0$ m/s^2, and on our moon, $g \approx 1.60$ m/s^2.

2.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative of a product is the product of the derivatives.
2. If $F(x) = f(x)g(x)$, then $F'(x) = $ _____.
3. *True or False* $\frac{d}{dx}x^n = nx^{n+1}$, for any integer n.
4. If f and $g \neq 0$ are two differentiable functions, then $\frac{d}{dx}\frac{f(x)}{g(x)} = $ _____.
5. *True or False* $f(x) = \frac{e^x}{x^2}$ can be differentiated using the Quotient Rule or by writing $f(x) = \frac{e^x}{x^2} = x^{-2}e^x$ and using the Product Rule.
6. If $g \neq 0$ is a differentiable function, then $\frac{d}{dx}\frac{1}{g(x)} = $ _____.
7. If $f(x) = x$, then $f''(x) = $ _____.
8. When an object in rectilinear motion is modeled by the position function $s = s(t)$, then the acceleration a of the object at time t is given by $a = a(t) = $ _____.

Skill Building

In Problems 9–40, find the derivative of each function.

PAGE 195 9. $f(x) = xe^x$
10. $f(x) = x^2e^x$
11. $f(x) = x^2(x^3 - 1)$
12. $f(x) = x^4(x + 5)$
PAGE 196 13. $f(x) = (3x^2 - 5)(2x + 1)$
14. $f(x) = (3x - 2)(4x + 5)$
15. $s(t) = (2t^5 - t)(t^3 - 2t + 1)$
16. $F(u) = (u^4 - 3u^2 + 1)(u^2 - u + 2)$
17. $f(x) = (x^3 + 1)(e^x + 1)$
18. $f(x) = (x^2 + 1)(e^x + x)$
19. $g(s) = \frac{2s}{s + 1}$
20. $F(z) = \frac{z + 1}{2z}$
21. $G(u) = \frac{1 - 2u}{1 + 2u}$
22. $f(w) = \frac{1 - w^2}{1 + w^2}$
PAGE 197 23. $f(x) = \frac{4x^2 - 2}{3x + 4}$
24. $f(x) = \frac{-3x^3 - 1}{2x^2 + 1}$
PAGE 197 25. $f(w) = \frac{1}{w^3 - 1}$
26. $g(v) = \frac{1}{v^2 + 5v - 1}$
27. $s(t) = t^{-3}$
28. $G(u) = u^{-4}$
29. $f(x) = -\frac{4}{e^x}$
30. $f(x) = \frac{3}{4e^x}$
PAGE 198 31. $f(x) = \frac{10}{x^4} + \frac{3}{x^2}$
32. $f(x) = \frac{2}{x^5} - \frac{3}{x^3}$
33. $f(x) = 3x^3 - \frac{1}{3x^2}$
34. $f(x) = x^5 - \frac{5}{x^5}$
35. $s(t) = \frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3}$
36. $s(t) = \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}$
37. $f(x) = \frac{e^x}{x^2}$
38. $f(x) = \frac{x^2}{e^x}$
39. $f(x) = \frac{x^2 + 1}{xe^x}$
40. $f(x) = \frac{xe^x}{x^2 - x}$

In Problems 41–54, find f' and f'' for each function.

PAGE 199 41. $f(x) = 3x^2 + x - 2$
42. $f(x) = -5x^2 - 3x$
43. $f(x) = e^x - 3$
44. $f(x) = x - e^x$
PAGE 200 45. $f(x) = (x + 5)e^x$
46. $f(x) = 3x^4e^x$
47. $f(x) = (2x + 1)(x^3 + 5)$
48. $f(x) = (3x - 5)(x^2 - 2)$
49. $f(x) = x + \frac{1}{x}$
50. $f(x) = x - \frac{1}{x}$
51. $f(t) = \frac{t^2 - 1}{t}$
52. $f(u) = \frac{u + 1}{u}$
53. $f(x) = \frac{e^x + x}{x}$
54. $f(x) = \frac{e^x}{x}$
55. Find y' and y'' for **(a)** $y = \frac{1}{x}$ and **(b)** $y = \frac{2x - 5}{x}$.
56. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for **(a)** $y = \frac{5}{x^2}$ and **(b)** $y = \frac{2 - 3x}{x}$.

Rectilinear Motion *In Problems 57–60, find the velocity $v = v(t)$ and acceleration $a = a(t)$ of an object in rectilinear motion whose signed distance s from the origin at time t is modeled by the position function $s = s(t)$.*

57. $s(t) = 16t^2 + 20t$ **58.** $s(t) = 16t^2 + 10t + 1$

59. $s(t) = 4.9t^2 + 4t + 4$ **60.** $s(t) = 4.9t^2 + 5t$

In Problems 61–68, find the indicated derivative.

61. $f^{(4)}(x)$ if $f(x) = x^3 - 3x^2 + 2x - 5$

62. $f^{(5)}(x)$ if $f(x) = 4x^3 + x^2 - 1$

63. $\dfrac{d^8}{dt^8}\left(\dfrac{1}{8}t^8 - \dfrac{1}{7}t^7 + t^5 - t^3\right)$ **64.** $\dfrac{d^6}{dt^6}(t^6 + 5t^5 - 2t + 4)$

65. $\dfrac{d^7}{du^7}(e^u + u^2)$ **66.** $\dfrac{d^{10}}{du^{10}}(2e^u)$

67. $\dfrac{d^5}{dx^5}(-e^x)$ **68.** $\dfrac{d^8}{dx^8}(12x - e^x)$

In Problems 69–72:

(a) *Find the slope of the tangent line for each function f at the given point.*

(b) *Find an equation of the tangent line to the graph of each function f at the given point.*

(c) *Find the points, if any, where the graph of the function has a horizontal tangent line.*

(d) *Graph each function, the tangent line found in (b), and any tangent lines found in (c) on the same set of axes.*

69. $f(x) = \dfrac{x^2}{x-1}$ at $\left(-1, -\dfrac{1}{2}\right)$ **70.** $f(x) = \dfrac{x}{x+1}$ at $(0, 0)$

71. $f(x) = \dfrac{x^3}{x+1}$ at $\left(1, \dfrac{1}{2}\right)$ **72.** $f(x) = \dfrac{x^2+1}{x}$ at $\left(2, \dfrac{5}{2}\right)$

In Problems 73–80:

(a) *Find the points, if any, at which the graph of each function f has a horizontal tangent line.*

(b) *Find an equation for each horizontal tangent line.*

(c) *Solve the inequality $f'(x) > 0$.*

(d) *Solve the inequality $f'(x) < 0$.*

(e) *Graph f and any horizontal lines found in (b) on the same set of axes.*

(f) *Describe the graph of f for the results obtained in (c) and (d).*

73. $f(x) = (x+1)(x^2 - x - 11)$ **74.** $f(x) = (3x^2 - 2)(2x + 1)$

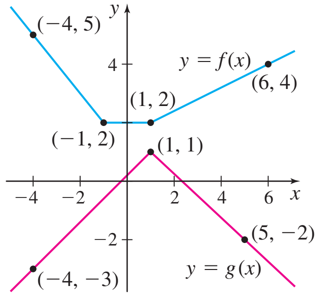
75. $f(x) = \dfrac{x^2}{x+1}$ **76.** $f(x) = \dfrac{x^2+1}{x}$

77. $f(x) = xe^x$ **78.** $f(x) = x^2e^x$

79. $f(x) = \dfrac{x^2-3}{e^x}$ **80.** $f(x) = \dfrac{e^x}{x^2+1}$

In Problems 81 and 82, use the graphs to determine each derivative.

81. Let $u(x) = f(x) \cdot g(x)$ and $v(x) = \dfrac{g(x)}{f(x)}$.

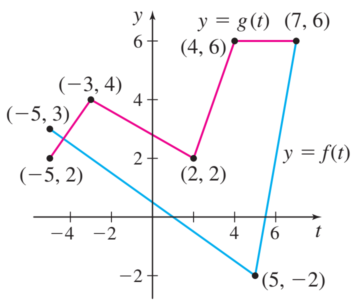


(a) $u'(0)$ **(b)** $u'(4)$

(c) $v'(-2)$ **(d)** $v'(6)$

(e) $\dfrac{d}{dx}\dfrac{1}{f(x)}$ at $x = -2$ **(f)** $\dfrac{d}{dx}\dfrac{1}{g(x)}$ at $x = 4$

82. Let $F(t) = f(t) \cdot g(t)$ and $G(t) = \dfrac{f(t)}{g(t)}$.



(a) $F'(0)$ **(b)** $F'(3)$

(c) $F'(-4)$ **(d)** $G'(-2)$

(e) $G'(-1)$ **(f)** $\dfrac{d}{dt}\dfrac{1}{f(t)}$ at $t = 3$

Applications and Extensions

PAGE 201

83. Vertical Motion An object is propelled vertically upward from the ground with an initial velocity of 39.2 m/s. The distance s (in meters) of the object from the ground after t seconds is given by the position function $s = s(t) = -4.9t^2 + 39.2t$.

(a) What is the velocity of the object at time t?

(b) When will the object reach its maximum height?

(c) What is the maximum height?

(d) What is the acceleration of the object at any time t?

(e) How long is the object in the air?

(f) What is the velocity of the object upon impact with the ground? What is its speed?

(g) What is the total distance traveled by the object?

84. Vertical Motion A ball is thrown vertically upward from a height of 6 ft with an initial velocity of 80 ft/s. The distance s (in feet) of the ball from the ground after t seconds is given by the position function $s = s(t) = 6 + 80t - 16t^2$.

(a) What is the velocity of the ball after 2 s?
(b) When will the ball reach its maximum height?
(c) What is the maximum height the ball reaches?
(d) What is the acceleration of the ball at any time t?
(e) How long is the ball in the air?
(f) What is the velocity of the ball upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the ball?

85. Environmental Cost The cost C, in thousands of dollars, for the removal of a pollutant from a certain lake is given by the function $C(x) = \dfrac{5x}{110 - x}$, where x is the percent of pollutant removed.

(a) What is the domain of C?
(b) Graph C.
(c) What is the cost to remove 80% of the pollutant?
(d) Find $C'(x)$, the rate of change of the cost C with respect to the amount of pollutant removed.
(e) Find the rate of change of the cost for removing 40%, 60%, 80%, and 90% of the pollutant.
(f) Interpret the answers found in (e).

86. Investing in Fine Art The value V of a painting t years after it is purchased is modeled by the function

$$V(t) = \frac{100t^2 + 50}{t} + 400 \quad 1 \le t \le 5$$

(a) Find the rate of change in the value V with respect to time.
(b) What is the rate of change in value after 2 years?
(c) What is the rate of change in value after 3 years?
(d) Interpret the answers in (b) and (c).

87. Drug Concentration The concentration of a drug in a patient's blood t hours after injection is given by the function $f(t) = \dfrac{0.4t}{2t^2 + 1}$ (in milligrams per liter).

(a) Find the rate of change of the concentration with respect to time.
(b) What is the rate of change of the concentration after 10 min? After 30 min? After 1 hour?
(c) Interpret the answers found in (b).
(d) Graph f for the first 5 hours after administering the drug.
(e) From the graph, approximate the time (in minutes) at which the concentration of the drug is highest. What is the highest concentration of the drug in the patient's blood?

88. Population Growth A population of 1000 bacteria is introduced into a culture and grows in number according to the formula

$$P(t) = 1000\left(1 + \frac{4t}{100 + t^2}\right), \text{ where } t \text{ is measured in hours.}$$

(a) Find the rate of change in population with respect to time.
(b) What is the rate of change in population at $t = 1$, $t = 2$, $t = 3$, and $t = 4$?
(c) Interpret the answers found in (b).
(d) Graph $P = P(t)$, $0 \le t \le 20$.
(e) From the graph, approximate the time (in hours) when the population is the greatest. What is the maximum population of the bacteria in the culture?

89. Economics The price-demand function for a popular e-book is given by $D(p) = \dfrac{100,000}{p^2 + 10p + 50}$, $4 \le p \le 20$, where $D = D(p)$ is the quantity demanded at the price p dollars.

(a) Find $D'(p)$, the rate of change of demand with respect to price.
(b) Find $D'(5)$, $D'(10)$, and $D'(15)$.
(c) Interpret the results found in (b).

90. Intensity of Light The intensity of illumination I on a surface is inversely proportional to the square of the distance r from the surface to the source of light. If the intensity is 1000 units when the distance is 1 m from the light, find the rate of change of the intensity with respect to the distance when the source is 10 meters from the surface.

PAGE 198 **91. Ideal Gas Law** The Ideal Gas Law, used in chemistry and thermodynamics, relates the pressure p, the volume V, and the absolute temperature T (in Kelvin) of a gas, using the equation $pV = nRT$, where n is the amount of gas (in moles) and $R = 8.31$ is the ideal gas constant. In an experiment, a spherical gas container of radius r meters is placed in a pressure chamber and is slowly compressed while keeping its temperature at 273 K.

(a) Find the rate of change of the pressure p with respect to the radius r of the chamber.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) If the sphere contains 1.0 mol of gas, find the rate of change of the pressure when $r = \frac{1}{4}$ m.
Note: The metric unit of pressure is the pascal, Pa.

92. Body Density The density ρ of an object is its mass m divided by its volume V; that is, $\rho = \dfrac{m}{V}$. If a person dives below the surface of the ocean, the water pressure on the diver will steadily increase, compressing the diver and therefore increasing body density. Suppose the diver is modeled as a sphere of radius r.

(a) Find the rate of change of the diver's body density with respect to the radius r of the sphere.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of the diver's body density when the radius is 45 cm and the mass is 80,000 g (80 kg).

Jerk and Snap *Problems 93–96 use the following discussion:* Suppose that an object is moving in rectilinear motion so that its signed distance s from the origin at time t is given by the position function $s = s(t)$. The velocity $v = v(t)$ of the object at time t is the rate of change of s with respect to time, namely, $v = v(t) = \dfrac{ds}{dt}$. The acceleration $a = a(t)$

of the object at time t is the rate of change of the velocity with respect to time,

$$a = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

There are also physical interpretations of the third derivative and the fourth derivative of $s = s(t)$. The **jerk** $J = J(t)$ of the object at time t is the rate of change of the acceleration a with respect to time; that is,

$$J = J(t) = \frac{da}{dt} = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The **snap** $S = S(t)$ of the object at time t is the rate of change of the jerk J with respect to time; that is,

$$S = S(t) = \frac{dJ}{dt} = \frac{d^2a}{dt^2} = \frac{d^3v}{dt^3} = \frac{d^4s}{dt^4}$$

Engineers take jerk into consideration when designing elevators, aircraft, and cars. In these cases, they try to minimize jerk, making for a smooth ride. But when designing thrill rides, such as roller coasters, the jerk is increased, making for an exciting experience.

93. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = t^3 - t + 1$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at time t.

(b) When is the velocity of the object 0 m/s?

(c) Find the acceleration of the object at $t = 2$ and at $t = 5$.

(d) Does the jerk of the object ever equal 0 m/s^3?

(e) How would you interpret the snap for this object in rectilinear motion?

94. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = \frac{1}{6}t^4 - t^2 + \frac{1}{2}t + 4$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at any time t.

(b) Find the velocity of the object at $t = 0$ and at $t = 3$.

(c) Find the acceleration of the object at $t = 0$. Interpret your answer.

(d) Is the jerk of the object constant? In your own words, explain what the jerk says about the acceleration of the object.

(e) How would you interpret the snap for this object in rectilinear motion?

95. Elevator Ride Quality The ride quality of an elevator depends on several factors, two of which are acceleration and jerk. In a study of 367 persons riding in a 1600-kg elevator that moves at an average speed of 4 m/s, the majority of riders were comfortable in an elevator with vertical motion given by

$$s(t) = 4t + 0.8t^2 + 0.333t^3$$

(a) Find the acceleration that the riders found acceptable.

(b) Find the jerk that the riders found acceptable.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

96. Elevator Ride Quality In a hospital, the effects of high acceleration or jerk may be harmful to patients, so the acceleration and jerk need to be lower than in standard elevators. It has been determined that a 1600-kg elevator that is installed in a hospital and that moves at an average speed of 4 m/s should have vertical motion

$$s(t) = 4t + 0.55t^2 + 0.1167t^3$$

(a) Find the acceleration of a hospital elevator.

(b) Find the jerk of a hospital elevator.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

97. Current Density in a Wire The current density J in a wire is a measure of how much an electrical current is compressed as it flows through a wire and is modeled by the function $J(A) = \frac{I}{A}$, where I is the current (in amperes) and A is the cross-sectional area of the wire. In practice, current density, rather than merely current, is often important. For example, superconductors lose their superconductivity if the current density is too high.

(a) As current flows through a wire, it heats the wire, causing it to expand in area A. If a constant current is maintained in a cylindrical wire, find the rate of change of the current density J with respect to the radius r of the wire.

(b) Interpret the sign of the answer found in (a).

(c) Find the rate of change of current density with respect to the radius r when a current of 2.5 amps flows through a wire of radius $r = 0.50$ mm.

98. Derivative of a Reciprocal, Function Prove that if a function g is differentiable, then $\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{[g(x)]^2}$, provided $g(x) \neq 0$.

99. Extended Product Rule Show that if f, g, and h are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

From this, deduce that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

In Problems 100–105, use the Extended Product Rule (Problem 99) to find y'.

100. $y = (x^2 + 1)(x - 1)(x + 5)$

101. $y = (x - 1)(x^2 + 5)(x^3 - 1)$

102. $y = (x^4 + 1)^3$

103. $y = (x^3 + 1)^3$

104. $y = (3x + 1)\left(1 + \frac{1}{x}\right)(x^{-5} + 1)$

105. $y = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)\left(1 - \frac{1}{x^3}\right)$

106. (Further) Extended Product Rule Write a formula for the derivative of the product of four differentiable functions. That is, find a formula for $\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)]$. Also find a formula for $\frac{d}{dx}[f(x)]^4$.

107. If f and g are differentiable functions, show that

if $F(x)=\dfrac{1}{f(x)g(x)}$, then

$$F'(x)=-F(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right]$$

provided $f(x)\neq 0$, $g(x)\neq 0$.

108. Higher-Order Derivatives If $f(x)=\dfrac{1}{1-x}$, find a formula for the nth derivative of f. That is, find $f^{(n)}(x)$.

109. Let $f(x)=\dfrac{x^6-x^4+x^2}{x^4+1}$. Rewrite f in the form $(x^4+1)f(x)=x^6-x^4+x^2$. Now find $f'(x)$ without using the quotient rule.

110. If f and g are differentiable functions with $f\neq -g$, find the derivative of $\dfrac{fg}{f+g}$.

CAS **111.** $f(x)=\dfrac{2x}{x+1}$.

(a) Use technology to find $f'(x)$.

(b) Simplify f' to a single fraction using either algebra or a CAS.

(c) Use technology to find $f^{(5)}(x)$.
Hint: Your CAS may have a method for finding higher-order derivatives without finding other derivatives first.

Challenge Problems

112. Suppose f and g have derivatives up to the fourth order. Find the first four derivatives of the product fg and simplify the answers. In particular, show that the fourth derivative is

$$\frac{d^4}{dx^4}(fg)=f^{(4)}g+4f^{(3)}g^{(1)}+6f^{(2)}g^{(2)}+4f^{(1)}g^{(3)}+fg^{(4)}$$

Identify a pattern for the higher-order derivatives of fg.

113. Suppose $f_1(x),\dots,f_n(x)$ are differentiable functions.

(a) Find $\dfrac{d}{dx}[f_1(x)\cdot\ldots\cdot f_n(x)]$.

(b) Find $\dfrac{d}{dx}\dfrac{1}{f_1(x)\cdot\ldots\cdot f_n(x)}$.

114. Let a, b, c, and d be real numbers. Define

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

This is called a 2×2 **determinant** and it arises in the study of linear equations. Let $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ be differentiable and let

$$D(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{vmatrix}$$

Show that

$$D'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) \\ f_3(x) & f_4(x) \end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x) \\ f_3'(x) & f_4'(x) \end{vmatrix}$$

115. Let $f_0(x)=x-1$

$$f_1(x)=1+\frac{1}{x-1}$$

$$f_2(x)=1+\cfrac{1}{1+\cfrac{1}{x-1}}$$

$$f_3(x)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x-1}}}$$

(a) Write f_1, f_2, f_3, f_4, and f_5 in the form $\dfrac{ax+b}{cx+d}$.

(b) Using the results from (a), write the sequence of numbers representing the coefficients of x in the numerator, beginning with $f_0(x)=x-1$.

(c) Write the sequence in (b) as a recursive sequence.
Hint: Look at the sum of consecutive terms.

(d) Find f_0', f_1', f_2', f_3', f_4', and f_5'.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 197 **1.** What is the instantaneous rate of change at $x=-2$ of the function $f(x)=\dfrac{x-1}{x^2+2}$?

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{9}$ (C) $\dfrac{1}{2}$ (D) -1

PAGE 197 **2.** An equation of the tangent line to the graph of $f(x)=\dfrac{5x-3}{3x-6}$ at the point $(3,4)$ is

(A) $7x+3y=37$ (B) $7x+3y=33$
(C) $7x-3y=9$ (D) $13x+3y=51$

PAGE 197 **3.** If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right)=$

(A) $\dfrac{fgh'+fg'h-f'gh}{f^2}$ (B) $\dfrac{g'h'-ghf'}{f^2}$

(C) $\dfrac{gh'+g'h}{f'}$ (D) $\dfrac{fgh'+fg'h+f'gh}{f^2}$

PAGE 195 **4.** If $y=x^3e^x$, then $\dfrac{dy}{dx}=$

(A) $3x^2e^x$ (B) $3x^2+e^x$
(C) $3x^2e^x(x+1)$ (D) $x^2e^x(x+3)$

PAGE 198 **5.** $\dfrac{d}{dt}\left(t^2-\dfrac{1}{t^2}+\dfrac{1}{t}\right)$ at $t=2$ is

(A) $\dfrac{7}{2}$ (B) $\dfrac{9}{2}$ (C) $\dfrac{9}{4}$ (D) 4

PAGE 201 **6.** The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s^2 (C) 32 m/s^2 (D) 64 m/s^2

PAGE 197 **7.** If $y = \dfrac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{6}$ (C) $\dfrac{1}{36}$ (D) 1

PAGE 197 **8.** Find an equation of the normal line to the graph of the function $f(x) = \dfrac{x^2}{x+1}$ at $x = 1$.

(A) $8x + 6y = 11$ (B) $-8x + 6y = -5$
(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

PAGE 200 **9.** If $y = xe^x$, then the nth derivative of y is

(A) e^x (B) $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

2.5 The Derivative of the Trigonometric Functions

OBJECTIVE ***When you finish this section, you should be able to:***

1 Differentiate trigonometric functions (p. 207)

1 Differentiate Trigonometric Functions

To find the derivatives of $y = \sin x$ and $y = \cos x$, we use the limits

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \text{and} \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

that were established in Section 1.4.

THEOREM Derivative of $y = \sin x$

The derivative of $y = \sin x$ is $y' = \cos x$. That is,

$$\boxed{y' = \frac{d}{dx} \sin x = \cos x}$$

Proof

$$y' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{The definition of a derivative}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \qquad \sin(A+B) = \sin A \cos B + \sin B \cos A$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right] \qquad \text{Rearrange terms.}$$

$$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cdot \cos x \right] \qquad \text{Factor.}$$

$$= \left[\lim_{h \to 0} \sin x \right] \left[\lim_{h \to 0} \frac{\cos h - 1}{h} \right] + \left[\lim_{h \to 0} \cos x \right] \left[\lim_{h \to 0} \frac{\sin h}{h} \right] \qquad \text{Use properties of limits.}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0; \quad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \blacksquare$$

NEED TO REVIEW? The trigonometric functions are discussed in Section P.6, pp. 52–58. Trigonometric identities are discussed in Appendix A.4, pp. A33 to A36.

The geometry of the derivative $\dfrac{d}{dx} \sin x = \cos x$ is shown in Figure 30. On the graph of $f(x) = \sin x$, the horizontal tangents are marked as well as the tangent lines that have slopes of 1 and -1. The derivative function is plotted on the second graph, and those points are connected with a smooth curve.

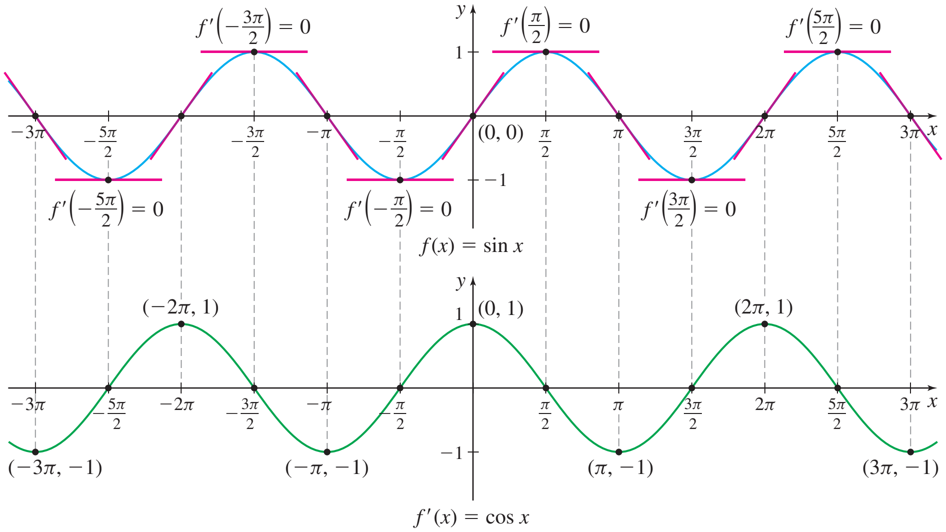


Figure 30

To find derivatives involving the trigonometric functions, use the sum, difference, product, and quotient rules and the derivative formulas from Sections 2.3 and 2.4.

EXAMPLE 1 Differentiating the Sine Function

Find y' if:

(a) $y = x + 4\sin x$ **(b)** $y = x^2 \sin x$ **(c)** $y = \dfrac{\sin x}{x}$ **(d)** $y = e^x \sin x$

Solution

(a) Use the Sum Rule and the Constant Multiple Rule.

$$y' = \frac{d}{dx}(x + 4\sin x) = \frac{d}{dx}x + \frac{d}{dx}(4\sin x) = 1 + 4\frac{d}{dx}\sin x = 1 + 4\cos x$$

(b) Use the Product Rule.

$$y' = \frac{d}{dx}(x^2 \sin x) = x^2\left[\frac{d}{dx}\sin x\right] + \left[\frac{d}{dx}x^2\right]\sin x = x^2\cos x + 2x\sin x$$

(c) Use the Quotient Rule.

$$y' = \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{\left[\frac{d}{dx}\sin x\right]\cdot x - \sin x \cdot \left[\frac{d}{dx}x\right]}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

(d) Use the Product Rule.

$$y' = \frac{d}{dx}(e^x \sin x) = e^x\frac{d}{dx}\sin x + \left(\frac{d}{dx}e^x\right)\sin x$$
$$= e^x\cos x + e^x \sin x = e^x(\cos x + \sin x)$$

■

NOW WORK Problems 5, 29 and AP® Practice Problems 1, 5 and 10.

THEOREM Derivative of $y = \cos x$

The derivative of $y = \cos x$ is

$$\boxed{y' = \frac{d}{dx}\cos x = -\sin x}$$

You are asked to prove this in Problem 75.

CALC CLIP

EXAMPLE 2 Differentiating Trigonometric Functions

Find the derivative of each function:

(a) $f(x) = x^2 \cos x$ **(b)** $g(\theta) = \dfrac{\cos\theta}{1 - \sin\theta}$ **(c)** $F(t) = \dfrac{e^t}{\cos t}$

Solution

(a)
$$f'(x) = \frac{d}{dx}(x^2 \cos x) = x^2 \frac{d}{dx}\cos x + \left(\frac{d}{dx}x^2\right)(\cos x)$$
$$= x^2(-\sin x) + 2x\cos x = 2x\cos x - x^2 \sin x$$

(b)
$$g'(\theta) = \frac{d}{d\theta}\left(\frac{\cos\theta}{1-\sin\theta}\right) = \frac{\left(\frac{d}{d\theta}\cos\theta\right)(1-\sin\theta) - (\cos\theta)\left[\frac{d}{d\theta}(1-\sin\theta)\right]}{(1-\sin\theta)^2}$$
$$= \frac{-\sin\theta\,(1-\sin\theta) - \cos\theta(-\cos\theta)}{(1-\sin\theta)^2} = \frac{-\sin\theta + \sin^2\theta + \cos^2\theta}{(1-\sin\theta)^2}$$
$$= \frac{-\sin\theta + 1}{(1-\sin\theta)^2} = \frac{1}{1-\sin\theta}$$

(c)
$$F'(t) = \frac{d}{dt}\left(\frac{e^t}{\cos t}\right) = \frac{\left(\frac{d}{dt}e^t\right)(\cos t) - e^t\left(\frac{d}{dt}\cos t\right)}{\cos^2 t} = \frac{e^t\cos t - e^t(-\sin t)}{\cos^2 t}$$
$$= \frac{e^t(\cos t + \sin t)}{\cos^2 t}$$
■

NOW WORK Problem 13 and AP Practice® Problems 2, 6 and 8.

EXAMPLE 3 Identifying Horizontal Tangent Lines

Find all points on the graph of $f(x) = x + \sin x$ where the tangent line is horizontal.

Solution

Since tangent lines are horizontal at points on the graph of f where $f'(x) = 0$, begin by finding $f'(x) = 1 + \cos x$. Now solve the equation:

$$f'(x) = 1 + \cos x = 0$$
$$\cos x = -1$$
$$x = (2k+1)\pi$$

where k is an integer.

Since $\sin[(2k+1)\pi] = 0$, then $f((2k+1)\pi) = (2k+1)\pi$. So, at each of the points $((2k+1)\pi, (2k+1)\pi)$, the graph of f has a horizontal tangent line. See Figure 31. ■

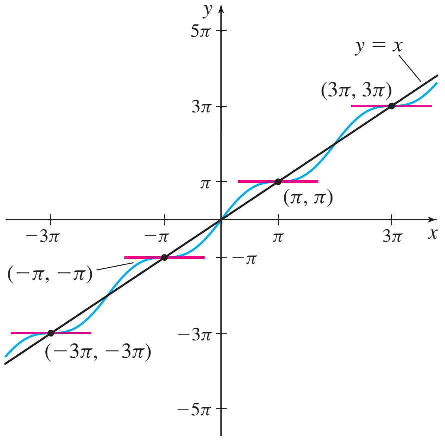


Figure 31 $f(x) = x + \sin x$

Notice in Figure 31 that each of the points with a horizontal tangent line lies on the line $y = x$.

NOW WORK Problem 57 and AP® Practice Problem 9.

The derivatives of the remaining four trigonometric functions are obtained using trigonometric identities and basic derivative rules. We establish the formula for the derivative of $y = \tan x$ in Example 4. You are asked to prove formulas for the derivative of the secant function, the cosecant function, and the cotangent function in the exercises. (See Problems 76–78.)

EXAMPLE 4 Differentiating $y = \tan x$

Show that the derivative of $y = \tan x$ is

$$\boxed{y' = \frac{d}{dx}\tan x = \sec^2 x}$$

Solution

$$y' = \frac{d}{dx}\tan x \underset{\uparrow \text{ Identity}}{=} \frac{d}{dx}\frac{\sin x}{\cos x} \underset{\uparrow \text{ Quotient Rule}}{=} \frac{\left[\frac{d}{dx}\sin x\right]\cos x - \sin x\left[\frac{d}{dx}\cos x\right]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$ ■

NOW WORK Problem 15 and AP® Practice Problems 3 and 7.

Table 4 lists the derivatives of the six trigonometric functions along with the domain of each derivative.

TABLE 4

Derivative Function	Domain of the Derivative Function
$\frac{d}{dx}\sin x = \cos x$	$(-\infty, \infty)$
$\frac{d}{dx}\cos x = -\sin x$	$(-\infty, \infty)$
$\frac{d}{dx}\tan x = \sec^2 x$	$\left\{x \mid x \neq \frac{2k+1}{2}\pi,\ k \text{ an integer}\right\}$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\{x \mid x \neq k\pi,\ k \text{ an integer}\}$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\{x \mid x \neq k\pi,\ k \text{ an integer}\}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\left\{x \mid x \neq \frac{2k+1}{2}\pi,\ k \text{ an integer}\right\}$

NOTE If the trigonometric function begins with the letter c, that is, cosine, cotangent, or cosecant, then its derivative has a minus sign.

NOW WORK Problem 35.

EXAMPLE 5 Finding the Second Derivative of a Trigonometric Function

Find $f''\left(\frac{\pi}{4}\right)$ if $f(x) = \sec x$.

Solution

If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$ and

$$f''(x) = \frac{d}{dx}(\sec x \tan x) \underset{\uparrow \text{ Use the Product Rule.}}{=} \sec x\left(\frac{d}{dx}\tan x\right) + \left(\frac{d}{dx}\sec x\right)\tan x$$

$$= \sec x \cdot \sec^2 x + (\sec x \tan x)\tan x = \sec^3 x + \sec x \tan^2 x$$

$$f''\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) \underset{\uparrow\ \sec\frac{\pi}{4} = \sqrt{2};\ \tan\frac{\pi}{4} = 1}{=} \left(\sqrt{2}\right)^3 + \sqrt{2}\cdot 1^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$$ ■

NOW WORK Problem 45 and AP® Practice Problem 4.

Application: Simple Harmonic Motion

Simple harmonic motion is a repetitive motion that can be modeled by a trigonometric function. A swinging pendulum and an oscillating spring are examples of simple harmonic motion.

EXAMPLE 6 Analyzing Simple Harmonic Motion

An object hangs on a spring, making the spring 2 m long in its equilibrium position. See Figure 32. If the object is pulled down 1 m and released, it oscillates up and down. The length l of the spring after t seconds is modeled by the function $l(t) = 2 + \cos t$.

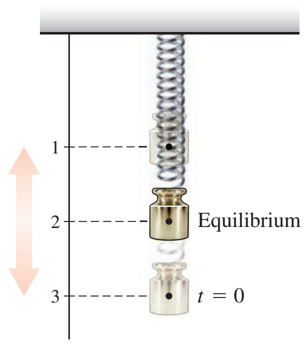


Figure 32

(a) How does the length of the spring vary?
(b) Find the velocity of the object.
(c) At what position is the speed of the object a maximum?
(d) Find the acceleration of the object.
(e) At what position is the acceleration equal to 0?

Solution

(a) Since $l(t) = 2 + \cos t$ and $-1 \le \cos t \le 1$, the length of the spring varies between 1 and 3 m.

(b) The velocity v of the object is

$$v = l'(t) = \frac{d}{dt}(2 + \cos t) = -\sin t$$

(c) Speed is the magnitude of velocity. Since $v = -\sin t$, the speed of the object is $|v| = |-\sin t| = |\sin t|$. Since $-1 \le \sin t \le 1$, the object moves the fastest when $|v| = |\sin t| = 1$. This occurs when $\sin t = \pm 1$ or, equivalently, when $\cos t = 0$. So, the speed is a maximum when $l(t) = 2$, that is, when the spring is at the equilibrium position.

(d) The acceleration a of the object is given by

$$a = l''(t) = \frac{d}{dt} l'(t) = \frac{d}{dt}(-\sin t) = -\cos t$$

(e) Since $a = -\cos t$, the acceleration is zero when $\cos t = 0$. So, $a = 0$ when $l(t) = 2$, that is, when the spring is at the equilibrium position. This is the same time at which the speed is maximum. ■

Figure 33 shows the graphs of the length of the spring $y = l(t)$, the velocity $y = v(t)$, and the acceleration $y = a(t)$.

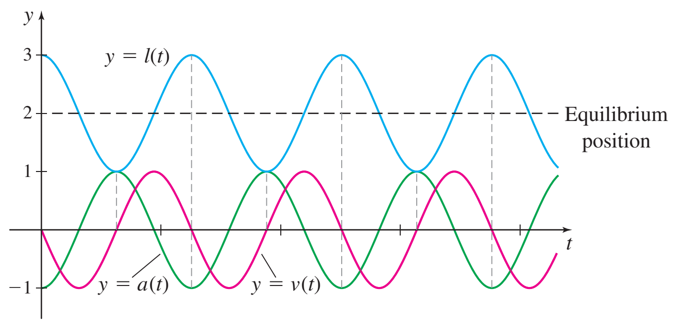


Figure 33 $y = l(t)$(blue), $y = v(t)$(red), $y = a(t)$(green)

NOW WORK Problem 65.

2.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\dfrac{d}{dx}\cos x = \sin x$

2. *True or False* $\dfrac{d}{dx}\tan x = \cot x$

3. *True or False* $\dfrac{d^2}{dx^2}\sin x = -\sin x$

4. *True or False* $\dfrac{d}{dx}\sin\dfrac{\pi}{3} = \cos\dfrac{\pi}{3}$

Skill Building

In Problems 5–38, find y'.

PAGE 208 **5.** $y = x - \sin x$

6. $y = \cos x - x^2$

7. $y = \tan x + \cos x$

8. $y = \sin x - \tan x$

9. $y = 3\sin\theta - 2\cos\theta$

10. $y = 4\tan\theta + \sin\theta$

11. $y = \sin x \cos x$

12. $y = \cot x \tan x$

PAGE 209 **13.** $y = t\cos t$

14. $y = t^2 \tan t$

PAGE 210 **15.** $y = e^x \tan x$

16. $y = e^x \sec x$

17. $y = \pi \sec u \tan u$

18. $y = \pi u \tan u$

19. $y = \dfrac{\cot x}{x}$

20. $y = \dfrac{\csc x}{x}$

21. $y = x^2 \sin x$

22. $y = t^2 \tan t$

23. $y = t\tan t - \sqrt{3}\sec t$

24. $y = x\sec x + \sqrt{2}\cot x$

25. $y = \dfrac{\sin\theta}{1-\cos\theta}$

26. $y = \dfrac{x}{\cos x}$

27. $y = \dfrac{\sin t}{1+t}$

28. $y = \dfrac{\tan u}{1+u}$

PAGE 208 **29.** $y = \dfrac{\sin x}{e^x}$

30. $y = \dfrac{\cos x}{e^x}$

31. $y = \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$

32. $y = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$

33. $y = \dfrac{\sec t}{1 + t\sin t}$

34. $y = \dfrac{\csc t}{1 + t\cos t}$

PAGE 210 **35.** $y = \csc\theta\cot\theta$

36. $y = \tan\theta\cos\theta$

37. $y = \dfrac{1+\tan x}{1-\tan x}$

38. $y = \dfrac{\csc x - \cot x}{\csc x + \cot x}$

In Problems 39–50, find y''.

39. $y = \sin x$

40. $y = \cos x$

41. $y = \tan\theta$

42. $y = \cot\theta$

43. $y = t\sin t$

44. $y = t\cos t$

PAGE 210 **45.** $y = e^x \sin x$

46. $y = e^x \cos x$

47. $y = 2\sin u - 3\cos u$

48. $y = 3\sin u + 4\cos u$

49. $y = a\sin x + b\cos x$

50. $y = a\sec\theta + b\tan\theta$

In Problems 51–56:

(a) *Find an equation of the tangent line to the graph of f at the indicated point.*

(b) *Graph the function and the tangent line.*

51. $f(x) = \sin x$ at $(0, 0)$

52. $f(x) = \cos x$ at $\left(\dfrac{\pi}{3}, \dfrac{1}{2}\right)$

53. $f(x) = \tan x$ at $(0, 0)$

54. $f(x) = \tan x$ at $\left(\dfrac{\pi}{4}, 1\right)$

55. $f(x) = \sin x + \cos x$ at $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$

56. $f(x) = \sin x - \cos x$ at $\left(\dfrac{\pi}{4}, 0\right)$

In Problems 57–60:

(a) *Find all points on the graph of f where the tangent line is horizontal.*

(b) *Graph the function and the horizontal tangent lines on the interval $[-2\pi, 2\pi]$.*

PAGE 209 **57.** $f(x) = 2\sin x + \cos x$

58. $f(x) = \cos x - \sin x$

59. $f(x) = \sec x$

60. $f(x) = \csc x$

Applications and Extensions

In Problems 61 and 62, find the nth derivative of each function.

61. $f(x) = \sin x$

62. $f(\theta) = \cos\theta$

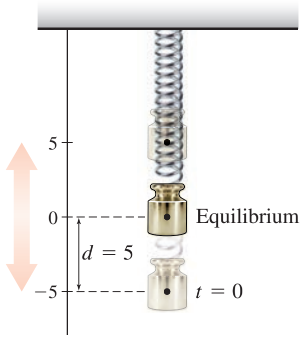
63. What is $\displaystyle\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\frac{\pi}{2}}{h}$?

64. What is $\displaystyle\lim_{h\to 0}\frac{\sin(\pi+h) - \sin\pi}{h}$?

PAGE 211 **65. Simple Harmonic Motion** The signed distance s (in meters) of an object from the origin at time t (in seconds) is modeled by the position function $s(t) = \dfrac{1}{8}\cos t$.

(a) Find the velocity $v = v(t)$ of the object.
(b) When is the speed of the object a maximum?
(c) Find the acceleration $a = a(t)$ of the object.
(d) When is the acceleration equal to 0?
(e) Graph s, v, and a on the same screen.

66. Simple Harmonic Motion An object attached to a coiled spring is pulled down a distance $d = 5$ cm from its equilibrium position and then released as shown in the figure. The motion of the object at time t seconds is simple harmonic and is modeled by $d(t) = -5\cos t$.



(a) As t varies from 0 to 2π, how does the length of the spring vary?

(b) Find the velocity $v = v(t)$ of the object.

(c) When is the speed of the object a maximum?

(d) Find the acceleration $a = a(t)$ of the object.

(e) When is the acceleration equal to 0?

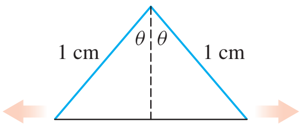
(f) Graph d, v, and a on the same set of axes.

67. Rate of Change A large, 8-ft-high decorative mirror is placed on a wood floor and leaned against a wall. The weight of the mirror and the slickness of the floor cause the mirror to slip.

(a) If θ is the angle between the top of the mirror and the wall, and y is the distance from the floor to the top of the mirror, what is the rate of change of y with respect to θ?

(b) In feet/radian, how fast is the top of the mirror slipping down the wall when $\theta = \dfrac{\pi}{4}$?

68. Rate of Change The sides of an isosceles triangle are sliding outward. See the figure.



(a) Find the rate of change of the area of the triangle with respect to θ.

(b) How fast is the area changing when $\theta = \dfrac{\pi}{6}$?

69. Sea Waves Waves in deep water tend to have the symmetric form of the function $f(x) = \sin x$. As they approach shore, however, the sea floor creates drag, which changes the shape of the wave. The trough of the wave widens and the height of the wave increases, so the top of the wave is no longer symmetric with the trough. This type of wave can be represented by a function such as

$$w(x) = \frac{4}{2 + \cos x}$$

(a) Graph $w = w(x)$ for $0 \le x \le 4\pi$.

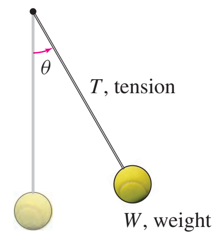
(b) What is the maximum and the minimum value of w?

(c) Find the values of x, $0 < x < 4\pi$, at which $w'(x) = 0$.

(d) Evaluate w' near the peak at π, using $x = \pi - 0.1$, and near the trough at 2π, using $x = 2\pi - 0.1$.

(e) Explain how these values confirm a nonsymmetric wave shape.

70. Swinging Pendulum A simple pendulum is a small-sized ball swinging from a light string. As it swings, the supporting string makes an angle θ with the vertical. See the figure. At an angle θ, the tension in the string is $T = \dfrac{W}{\cos\theta}$, where W is the weight of the swinging ball.



(a) Find the rate of change of the tension T with respect to θ when the pendulum is at its highest point ($\theta = \theta_{\max}$).

(b) Find the rate of change of the tension T with respect to θ when the pendulum is at its lowest point.

(c) What is the tension at the lowest point?

71. Restaurant Sales A restaurant in Naples, Florida, is very busy during the winter months and extremely slow over the summer. But every year the restaurant grows its sales. Suppose over the next two years, the revenue R, in units of \$10,000, is projected to follow the model

$$R = R(t) = \sin t + 0.3t + 1 \qquad 0 \le t \le 12$$

where $t = 0$ corresponds to November 1, 2018; $t = 1$ corresponds to January 1, 2019; $t = 2$ corresponds to March 1, 2019; and so on.

(a) What is the projected revenue for November 1, 2018; March 1, 2019; September 1, 2019; and January 1, 2020?

(b) What is the rate of change of revenue with respect to time?

(c) What is the rate of change of revenue with respect to time for January 1, 2020?

(d) Graph the revenue function and the derivative function $R' = R'(t)$.

(e) Does the graph of R support the facts that every year the restaurant grows its sales and that sales are higher during the winter and lower during the summer? Explain.

72. Polarizing Sunglasses Polarizing sunglasses are filters that transmit only light for which the electric field oscillations are in a specific direction. Light is polarized naturally by scattering off the molecules in the atmosphere and by reflecting off many (but not all) types of surfaces. If light of intensity I_0 is already polarized in a certain direction, and the transmission direction of the polarizing filter makes an angle with that direction, then the intensity I of the light after passing through the filter is given by **Malus's Law**, $I(\theta) = I_0\cos^2\theta$.

REUTERS/Alamy

(a) As you rotate a polarizing filter, θ changes. Find the rate of change of the light intensity I with respect to θ.

(b) Find both the intensity $I(\theta)$ and the rate of change of the intensity with respect to θ, for the angles $\theta = 0°, 45°$, and $90°$. (Remember to use radians for θ.)

73. If $y = \sin x$ and $y^{(n)}$ is the nth derivative of y with respect to x, find the smallest positive integer n for which $y^{(n)} = y$.

74. Use the identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$, with $A = x + h$ and $B = x$, to prove that

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$$

75. Use the definition of a derivative to prove $\frac{d}{dx}\cos x = -\sin x$.

76. Derivative of $y = \sec x$ Use a derivative rule to show that

$$\frac{d}{dx}\sec x = \sec x\tan x$$

77. Derivative of $y = \csc x$ Use a derivative rule to show that

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

78. Derivative of $y = \cot x$ Use a derivative rule to show that

$$\frac{d}{dx}\cot x = -\csc^2 x$$

79. Let $f(x) = \cos x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\cos x - 1}{x}$.

80. Let $f(x) = \sin x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\sin x}{x}$.

81. If $y = A\sin t + B\cos t$, where A and B are constants, show that $y'' + y = 0$.

Challenge Problem

82. For a differentiable function f, let f^* be the function defined by

$$f^*(x) = \lim_{h\to 0}\frac{f(x+h) - f(x-h)}{h}$$

(a) Find $f^*(x)$ for $f(x) = x^2 + x$.

(b) Find $f^*(x)$ for $f(x) = \cos x$.

(c) Write an equation that expresses the relationship between the functions f^* and f', where f' denotes the derivative of f. Justify your answer.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 208 **1.** If $y = x\sin x$, then $\frac{dy}{dx} =$

(A) $x\cos x + \sin x$ (B) $x\cos x - \sin x$
(C) $\cos x + \sin x$ (D) $(x+1)\cos x$

PAGE 209 **2.** What is $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3}+h\right) - \cos\frac{\pi}{3}}{h}$?

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$

PAGE 210 **3.** If $f(x) = \tan x$, then $f'\left(\frac{\pi}{3}\right)$ equals

(A) $2\sqrt{3}$ (B) 4 (C) 2 (D) $\frac{1}{4}$

PAGE 210 **4.** The position s (in meters) of an object moving along a horizontal line at time t, $0 \le t \le \frac{\pi}{2}$, (in seconds) is given by $s(t) = 6\sin t + \frac{3}{2}t^2 + 8$. What is the velocity of the object when its acceleration is zero?

(A) 6 m/s (B) $3 + \pi$ m/s
(C) $\frac{6\sqrt{3}+\pi}{2}$ m/s (D) $\left(3\sqrt{3} - \frac{\pi}{2}\right)$ m/s

PAGE 208 **5.** If $y = \sin x$, then $\frac{d^{50}}{dx^{50}}\sin x$ equals

(A) $\sin x$ (B) $-\sin x$ (C) $\cos x$ (D) $-\cos x$

PAGE 209 **6.** If $f(x) = \frac{x}{\cos x}$, find $f'\left(\frac{\pi}{3}\right)$.

(A) $2 - \frac{2\sqrt{3}}{3}\pi$ (B) $1 + \frac{\sqrt{3}}{3}\pi$
(C) $1 - \frac{\sqrt{3}}{3}\pi$ (D) $2 + \frac{2\sqrt{3}}{3}\pi$

PAGE 210 **7.** If $y = x - \tan x$, then $\frac{dy}{dx}$ equals

(A) $1 - \sec x\tan x$ (B) $-\tan^2 x$
(C) $\tan^2 x$ (D) $-\sec^2 x$

PAGE 209 **8.** If $g(x) = e^x\cos x + 2\pi$, then $g'(x) =$

(A) $e^x - \sin x$ (B) $e^x\cos x - e^x\sin x + 3\pi$
(C) $e^x\cos x - e^x\sin x$ (D) $e^x\cos x + e^x\sin x$

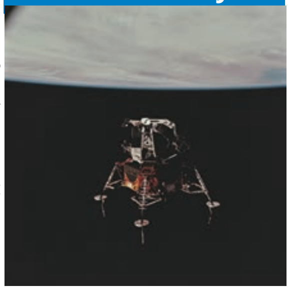
PAGE 209 **9.** At which of the following numbers x, $0 \le x \le 2\pi$, does the graph of $y = x + \cos x$ have a horizontal tangent line?

(A) 0 only (B) $\frac{\pi}{2}$ only
(C) $\frac{3\pi}{2}$ only (D) 0 and $\frac{\pi}{2}$ only

PAGE 208 **10.** An equation of the tangent line to the graph of $f(x) = \sin x$ at $x = \frac{2\pi}{3}$ is

(A) $3x + 6y = 4\pi - 3\sqrt{3}$ (B) $3x + 6y = 2\pi + 3\sqrt{3}$
(C) $6y - 3x = 2\pi - 3\sqrt{3}$ (D) $6y - 3x = 4\pi - 3\sqrt{3}$

CHAPTER 2 PROJECT The *Apollo* Lunar Module


Rolls Press/Popperfoto/Getty Images

The Lunar Module (LM) was a small spacecraft that detached from the *Apollo* Command Module and was designed to land on the Moon. Fast and accurate computations were needed to bring the LM from an orbiting speed of about 5500 ft/s to a speed slow enough to land it within a few feet of a designated target on the Moon's surface. The LM carried a 70-lb computer to assist in guiding it successfully to its target. The approach to the target was split into three phases, each of which followed a reference trajectory specified by NASA engineers.*
The position and velocity of the LM were monitored by sensors that tracked its deviation from the preassigned path at each moment. Whenever the LM strayed from the reference trajectory, control thrusters were fired to reposition it. In other words, the LM's position and velocity were adjusted by changing its acceleration.

The reference trajectory for each phase was specified by the engineers to have the form

$$r_{\text{ref}}(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_T t^3 + \frac{1}{24}S_T t^4 \qquad (1)$$

The variable r_{ref} represents the intended position of the LM at time t before the end of the landing phase. The engineers specified the end of the landing phase to take place at $t = 0$, so that during the phase, t was always negative. Note that the LM was landing in three dimensions, so there were actually three equations like (1). Since each of those equations had this same form, we will work in one dimension, assuming, for example, that r represents the distance of the LM above the surface of the Moon.

1. If the LM follows the reference trajectory, what is the reference velocity $v_{\text{ref}}(t)$?
2. What is the reference acceleration a (ref$t$)?
3. The rate of change of acceleration is called **jerk**. Find the reference jerk $J_{\text{ref}}(t)$.
4. The rate of change of jerk is called **snap**. Find the reference snap $S_{\text{ref}}(t)$.
5. Evaluate $r_{\text{ref}}(t)$, $v_{\text{ref}}(t)$, $a_{\text{ref}}(t)$, $J_{\text{ref}}(t)$, and $S_{\text{ref}}(t)$ when $t = 0$.

The reference trajectory given in equation (1) is a fourth-degree polynomial, the lowest degree polynomial that has enough free parameters to satisfy all the mission criteria. Now we see that the parameters $R_T = r_{\text{ref}}(0)$, $V_T = v_{\text{ref}}(0)$, $A_T = a_{\text{ref}}(0)$, $J_T = J_{\text{ref}}(0)$, and $S_T = S_{\text{ref}}(0)$. The five parameters in equation (1) are referred to as the **target parameters** since they provide the path the LM should follow.

But small variations in propulsion, mass, and countless other variables cause the LM to deviate from the predetermined path. To correct the LM's position and velocity, NASA engineers apply a force to the LM using rocket thrusters. That is, they changed the acceleration. (Remember Newton's second law, $F = ma$.) Engineers modeled the actual trajectory of the LM by

$$r(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_A t^3 + \frac{1}{24}S_A t^4 \qquad (2)$$

We know the target parameters for position, velocity, and acceleration. We need to find the actual parameters for jerk and snap to know the proper force (acceleration) to apply.

6. Find the actual velocity $v = v(t)$ of the LM.
7. Find the actual acceleration $a = a(t)$ of the LM.
8. Use equation (2) and the actual velocity found in Problem 6 to express J_A and S_A in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.
9. Use the results of Problems 7 and 8 to express the actual acceleration $a = a(t)$ in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.

The result found in Problem 9 provides the acceleration (force) required to keep the LM in its reference trajectory.

10. When riding in an elevator, the sensation one feels just before the elevator stops at a floor is jerk. Would you want jerk to be small or large in an elevator? Explain. Would you want jerk to be small or large on a roller coaster ride? Explain. How would you explain snap?

*A. R. Klumpp, "*Apollo* Lunar-Descent Guidance," MIT Charles Stark Draper Laboratory, R-695, June 1971, http://www.hq.nasa.gov/alsj/ApolloDescentGuidnce.pdf

Chapter Review

THINGS TO KNOW

2.1 Rates of Change and the Derivative

- **Definition** (Form 1) Derivative of a function f at a number c

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists. (p. 167)

Three Interpretations of the Derivative

- *Geometric* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$. (p. 167)
- *Rate of change of a function* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f with respect to x at c. (p. 167)
- *Physical* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s = f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0. (p. 167)

2.2 The Derivative as a Function

- **Definition of a derivative function** (Form 2)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. (p. 172)

- **Theorem** If a function f has a derivative at a number c, then f is continuous at c. (p. 177)
- **Corollary** If a function f is discontinuous at a number c, then f has no derivative at c. (p. 177)

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

- **Leibniz notation** $\dfrac{dy}{dx} = \dfrac{d}{dx}y = \dfrac{d}{dx}f(x)$ (p. 183)
- **Basic derivatives**

$\dfrac{d}{dx}A = 0$ A is a constant (p. 184) $\qquad \dfrac{d}{dx}x = 1$ (p. 184)

$\dfrac{d}{dx}e^x = e^x$ (p. 190) $\qquad \dfrac{d}{dx}\ln x = \dfrac{1}{x}$ (p. 190)

- *Simple Power Rule* $\dfrac{d}{dx}x^n = nx^{n-1}$, $n \geq 1$, an integer (p. 185)

Properties of Derivatives

- *Sum Rule* (pp. 186, 187) $\dfrac{d}{dx}[f+g] = \dfrac{d}{dx}f + \dfrac{d}{dx}g$

$$(f+g)' = f' + g'$$

- *Difference Rule* (p. 187) $\dfrac{d}{dx}[f-g] = \dfrac{d}{dx}f - \dfrac{d}{dx}g$

$$(f-g)' = f' - g'$$

- *Constant Multiple Rule* (p. 186) If k is a constant,

$$\frac{d}{dx}[kf] = k\frac{d}{dx}f$$
$$(kf)' = k \cdot f'$$

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

Properties of Derivatives

- *Product Rule* (p. 195) $\dfrac{d}{dx}(fg) = f\left(\dfrac{d}{dx}g\right) + \left(\dfrac{d}{dx}f\right)g$

$$(fg)' = fg' + f'g$$

- *Quotient Rule* (p. 196) $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\left(\dfrac{d}{dx}f\right)g - f\left(\dfrac{d}{dx}g\right)}{g^2}$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

provided $g(x) \neq 0$

- *Reciprocal Rule* (p. 197) $\dfrac{d}{dx}\left(\dfrac{1}{g}\right) = -\dfrac{\dfrac{d}{dx}g}{g^2}$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

provided $g(x) \neq 0$

- *Power Rule* $\dfrac{d}{dx}x^n = nx^{n-1}$, n an integer (p. 198)
- *Higher-order derivatives* See Table 3 (p. 199)
- *Position Function* $s = s(t)$ (p. 200)
- *Velocity* $v = v(t) = \dfrac{ds}{dt}$ (p. 200)
- *Acceleration* $a = a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$ (p. 200)

2.5 The Derivative of the Trigonometric Functions

Basic Derivatives

$\dfrac{d}{dx}\sin x = \cos x$ (p. 207) $\qquad \dfrac{d}{dx}\sec x = \sec x \tan x$ (p. 210)

$\dfrac{d}{dx}\cos x = -\sin x$ (p. 208) $\qquad \dfrac{d}{dx}\csc x = -\csc x \cot x$ (p. 210)

$\dfrac{d}{dx}\tan x = \sec^2 x$ (p. 210) $\qquad \dfrac{d}{dx}\cot x = -\csc^2 x$ (p. 210)

OBJECTIVES

Section	You should be able to ...	Examples	Review Exercises	Preparing for the AP® Exam AP® Review Problems
2.1	1 Find equations for the tangent line and the normal line to the graph of a function (p. 162)	1	67–70	
	2 Find the rate of change of a function (p. 163)	2, 3	1, 2, 73 (a)	
	3 Find average velocity and instantaneous velocity (p. 164)	4, 5	71(a), (b); 72(a), (b)	
	4 Find the derivative of a function at a number (p. 166)	6–8	3–8, 75	5
2.2	1 Define the derivative function (p. 171)	1–3	9–12, 77	2
	2 Graph the derivative function (p. 173)	4, 5	9–12, 15–18	
	3 Identify where a function is not differentiable (p. 175)	6–10	13, 14, 75	4
2.3	1 Differentiate a constant function (p. 184)	1		
	2 Differentiate a power function (p. 184)	2, 3	19–22	
	3 Differentiate the sum and the difference of two functions (p. 186)	4–6	23–26, 33, 34, 40, 51, 52, 67	
	4 Differentiate the exponential function $y=e^x$ (p. 189)	7	44, 45, 53, 54, 56, 59, 69	7
2.4	1 Differentiate the product of two functions (p. 194)	1, 2	27, 28, 36, 46, 48–50, 53–56, 60	
	2 Differentiate the quotient of two functions (p. 196)	3–6	29–35, 37–43, 47, 57–59, 68, 73, 74	3, 10
	3 Find higher-order derivatives (p. 198)	7, 8	61–66, 71, 72, 76	
	4 Find the acceleration of an object in rectilinear motion (p. 200)	9	71, 72, 76	8
2.5	1 Differentiate trigonometric functions (p. 207)	1–6	49–60, 70	1, 6, 9

REVIEW EXERCISES

In Problems 1 and 2, use a definition of the derivative to find the rate of change of f at the indicated numbers.

1. $f(x)=\sqrt{x}$ at **(a)** $c=1$ **(b)** $c=4$ 1, 2 **(c)** c any positive real number

2. $f(x)=\dfrac{2}{x-1}$ at **(a)** $c=0$ **(b)** $c=2$ **(c)** c any real number, $c\neq 1$

In Problems 3–8, use a definition of the derivative to find the derivative of each function at the given number.

3. $F(x)=2x+5$ at 2

4. $f(x)=4x^2+1$ at -1

5. $f(x)=3x^2+5x$ at 0

6. $f(x)=\dfrac{3}{x}$ at 1

7. $f(x)=\sqrt{4x+1}$ at 0

8. $f(x)=\dfrac{x+1}{2x-3}$ at 1

In Problems 9–12, use a definition of the derivative to find the derivative of each function. Graph f and f' on the same set of axes.

9. $f(x)=x-6$

10. $f(x)=7-3x^2$

11. $f(x)=\dfrac{1}{2x^3}$

12. $f(x)=\pi$

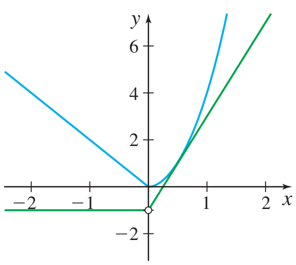
In Problems 13 and 14, determine whether the function f has a derivative at c. If it does, find the derivative. If it does not, explain why. Graph each function.

13. $f(x)=|x^3-1|$ at $c=1$

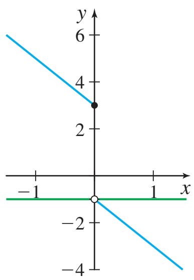
14. $f(x)=\begin{cases}4-3x^2 & \text{if } x\leq -1\\ -x^3 & \text{if } x>-1\end{cases}$ at $c=-1$

In Problems 15 and 16, determine whether the graphs represent a function f and its derivative f'. If they do, indicate which is the graph of f and which is the graph of f'.

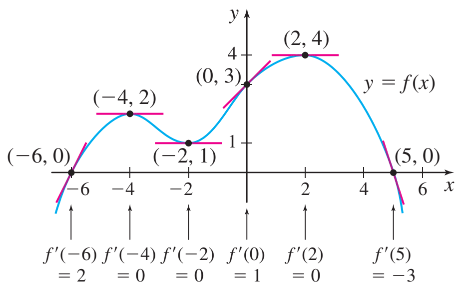
15.



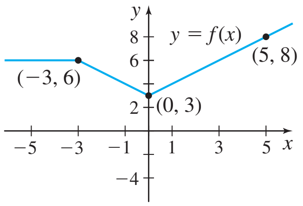
16.

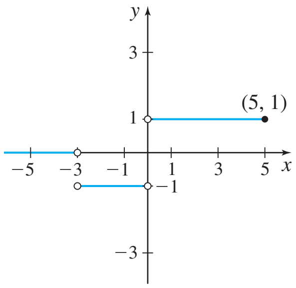


17. Use the information in the graph of $y=f(x)$ to sketch the graph of $y=f'(x)$.

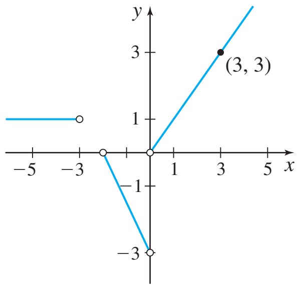


18. Match the graph of $y = f(x)$ with the graph of its derivative.

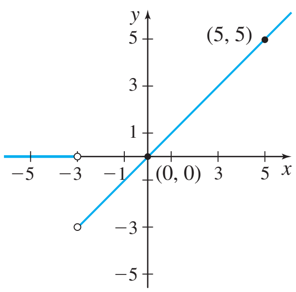




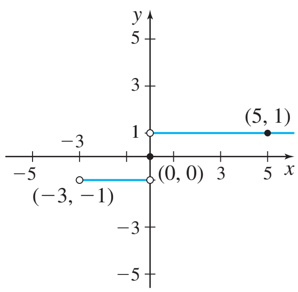
(A)



(B)



(C)



(D)

In Problems 19–60, find the derivative of each function. Treat a and b, if present, as constants.

19. $f(x) = x^5$

20. $f(x) = ax^3$

21. $f(x) = \dfrac{x^4}{4}$

22. $f(x) = -6x^2$

23. $f(x) = 2x^2 - 3x$

24. $f(x) = 3x^3 + \dfrac{2}{3}x^2 - 5x + 7$

25. $F(x) = 7(x^2 - 4)$

26. $F(x) = \dfrac{5(x+6)}{7}$

27. $f(x) = 5(x^2 - 3x)(x - 6)$

28. $f(x) = (2x^3 + x)(x^2 - 5)$

29. $f(x) = \dfrac{6x^4 - 9x^2}{3x^3}$

30. $f(x) = \dfrac{2x + 2}{5x - 3}$

31. $f(x) = \dfrac{7x}{x - 5}$

32. $f(x) = 2x^{-12}$

33. $f(x) = 2x^2 - 5x^{-2}$

34. $f(x) = 2 + \dfrac{3}{x} + \dfrac{4}{x^2}$

35. $f(x) = \dfrac{a}{x} - \dfrac{b}{x^3}$

36. $f(x) = (x^3 - 1)^2$

37. $f(x) = \dfrac{3}{(x^2 - 3x)^2}$

38. $f(x) = \dfrac{x^2}{x + 1}$

39. $s(t) = \dfrac{t^3}{t - 2}$

40. $f(x) = 3x^{-2} + 2x^{-1} + 1$

41. $F(z) = \dfrac{1}{z^2 + 1}$

42. $f(v) = \dfrac{v - 1}{v^2 + 1}$

43. $g(z) = \dfrac{1}{1 - z + z^2}$

44. $f(x) = 3e^x + x^2$

45. $s(t) = 1 - e^t$

46. $f(x) = ae^x(2x^2 + 7x)$

47. $f(x) = \dfrac{1 + x}{e^x}$

48. $f(x) = (2xe^x)^2$

49. $f(x) = x \sin x$

50. $s(t) = \cos^2 t$

51. $G(u) = \tan u + \sec u$

52. $g(v) = \sin v - \dfrac{1}{3}\cos v$

53. $f(x) = e^x \sin x$

54. $f(x) = e^x \csc x$

55. $f(x) = 2 \sin x \cos x$

56. $f(x) = (e^x + b)\cos x$

57. $f(x) = \dfrac{\sin x}{\csc x}$

58. $f(x) = \dfrac{1 - \cot x}{1 + \cot x}$

59. $f(\theta) = \dfrac{\cos\theta}{2e^\theta}$

60. $f(\theta) = 4\theta \cot\theta \tan\theta$

In Problems 61–66, find the first derivative and the second derivative of each function.

61. $f(x) = (5x + 3)^2$

62. $f(x) = xe^x$

63. $g(u) = \dfrac{u}{2u + 1}$

64. $F(x) = e^x(\sin x + 2\cos x)$

65. $f(u) = \dfrac{\cos u}{e^u}$

66. $F(x) = \dfrac{\sin x}{x}$

In Problems 67–70, for each function:

(a) *Find an equation of the tangent line to the graph of the function at the indicated point.*

(b) *Find an equation of the normal line to the function at the indicated point.*

(c) *Graph the function, the tangent line, and the normal line on the same screen.*

67. $f(x) = 2x^2 - 3x + 7$ at $(-1, 12)$

68. $y = \dfrac{x^2 + 1}{2x - 1}$ at $\left(2, \dfrac{5}{3}\right)$

69. $f(x) = x^2 - e^x$ at $(0, -1)$

70. $s(t) = 1 + 2\sin t$ at $(\pi, 1)$

71. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin at time t (in seconds) is given by the position function

$$s = f(t) = t^2 - 6t$$

(a) Find the average velocity of the object from 0 to 5 s.

(b) Find the velocity at $t = 0$, at $t = 5$, and at any time t.

(c) Find the acceleration at any time t.

72. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s(t) = t - t^2$, where s is in centimeters and t is in seconds.

(a) Find the average velocity of the object from 1 to 3 s.

(b) Find the velocity of the object at $t = 1$ s and $t = 3$ s.

(c) What is its acceleration at $t = 1$ and $t = 3$?

73. **Business** The price p in dollars per pound when x pounds of a commodity are demanded is modeled by the function

$$p(x) = \frac{10{,}000}{5x + 100} - 5$$

when between 0 and 90 lb are demanded (purchased).

(a) Find the rate of change of price with respect to demand.

(b) What is the revenue function R? (Recall, revenue R equals price times amount purchased.)

(c) What is the marginal revenue R' at $x = 10$ and at $x = 40$ lb?

74. If $f(x) = \dfrac{x-1}{x+1}$ for all $x \neq -1$, find $f'(1)$.

75. If $f(x) = 2 + |x - 3|$ for all x, determine whether the derivative f' exists at $x = 3$.

76. **Rectilinear Motion** An object in rectilinear motion moves according to the position function $s = 2t^3 - 15t^2 + 24t + 3$, where t is measured in minutes and s in meters.

(a) When is the object at rest?

(b) Find the object's acceleration when $t = 3$.

77. Find the value of the limit below and specify the function f for which this is the derivative.

$$\lim_{\Delta x \to 0} \frac{[4 - 2(x + \Delta x)]^2 - (4 - 2x)^2}{\Delta x}$$

Preparing for the AP® Exam

AP® REVIEW PROBLEMS: CHAPTER 2

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 (D) $\sqrt{2}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

I. $\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$

II. $\lim_{x\to c} \frac{f(x+h) - f(x)}{h}$

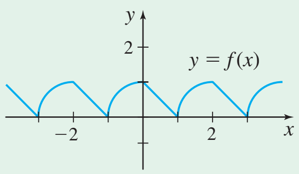
III. $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$

(A) I only (B) III only
(C) I and II only (D) I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

(A) $\frac{6x}{(x^2-5)^2}$ (B) $-\frac{6x}{(x^2-5)^2}$
(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere.

(B) $0 \le f'(x) \le 1$, for all real numbers.

(C) f is continuous everywhere.

(D) f is an even function.

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

(A) 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2
(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$
(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only
(C) at 3 only (D) at 0 and 3 only

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$
(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

10. Find an equation of the tangent line to the graph of $f(x) = \frac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$
(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

11. $\lim_{x\to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) −1 (C) 2 (D) Does not exist.

AP® CUMULATIVE REVIEW PROBLEMS: CHAPTERS 1–2

1. $\lim_{x\to 4} \dfrac{x-4}{4-x} =$

(A) -4 (B) -1 (C) 0 (D) does not exist

2. $\lim_{x\to 0} \dfrac{3x+\sin x}{2x} =$

(A) 0 (B) 1 (C) 2 (D) does not exist

3. Let h be defined by

$$h(x) = \begin{cases} f(x)\cdot g(x) & \text{if} \quad x \le 1 \\ k + x & \text{if} \quad x > 1 \end{cases}$$

where f and g are both continuous at all real numbers. If $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = -2$, then for what number k is h continuous?

(A) -5 (B) -4 (C) -2 (D) 2

4. Which function has the horizontal asymptotes $y = 1$ and $y = -1$?

(A) $f(x) = \dfrac{2}{\pi}\tan^{-1} x$ (B) $f(x) = e^{-x} + 1$

(C) $f(x) = \dfrac{1-x^2}{1+x^2}$ (D) $f(x) = \dfrac{2x^2-1}{2x^2+x}$

5. Suppose the function f is continuous at all real numbers and $f(-2) = 1$ and $f(5) = -3$. Suppose the function g is also continuous at all real numbers and $g(x) = f^{-1}(x)$ for all x. The Intermediate Value Theorem guarantees that

(A) $g(c) = 2$ for at least one c between -3 and 1.

(B) $g(c) = 0$ for at least one c between -2 and 5.

(C) $f(c) = 0$ for at least one c between -3 and 1.

(D) $f(c) = 2$ for at least one c between -2 and 5.

6. The line $x = c$ is a vertical asymptote to the graph of the function f. Which of the following statements cannot be true?

(A) $\lim_{x\to c} f(x) = \infty$ (B) $\lim_{x\to\infty} f(x) = c$

(C) $f(c)$ is not defined. (D) f is continuous at $x = c$.

7. The position function of an object moving along a straight line is $s(t) = \dfrac{1}{15}t^3 - \dfrac{1}{2}t^2 + 5t^{-1}$. What is the object's acceleration at $t = 5$?

(A) $-\dfrac{27}{25}$ (B) $-\dfrac{1}{5}$ (C) $\dfrac{1}{5}$ (D) $\dfrac{27}{25}$

8. If the function $f(x) = \begin{cases} 2ax^2 + bx - 1 & \text{if} \quad x \le 3 \\ bx^2 + bx - a & \text{if} \quad x > 3 \end{cases}$

is continuous for all real numbers x, then

(A) $19a - 15b = 1$ (B) $18a - 9b = 1$

(C) $19a - 9b = 1$ (D) $19a + 15b = 1$

9. Find the slope of the tangent line to the graph of $f(x) = xe^x$ at the point $(1, e)$.

(A) 1 (B) e (C) $2e$ (D) e^2

10. An object in rectilinear motion is modeled by the position function

$$s(t) = 3t^4 - 8t^3 - 6t^2 + 24t \qquad t > 0$$

where s is in feet (ft) and t is in seconds (s). Find the acceleration of the object when its velocity is zero.

(A) $-24\ \text{ft/s}^2$, $36\ \text{ft/s}^2$, and $72\ \text{ft/s}^2$ only

(B) $36\ \text{ft/s}^2$ only

(C) $36\ \text{ft/s}^2$ and $72\ \text{ft/s}^2$ only

(D) $-24\ \text{ft/s}^2$ and $36\ \text{ft/s}^2$ only

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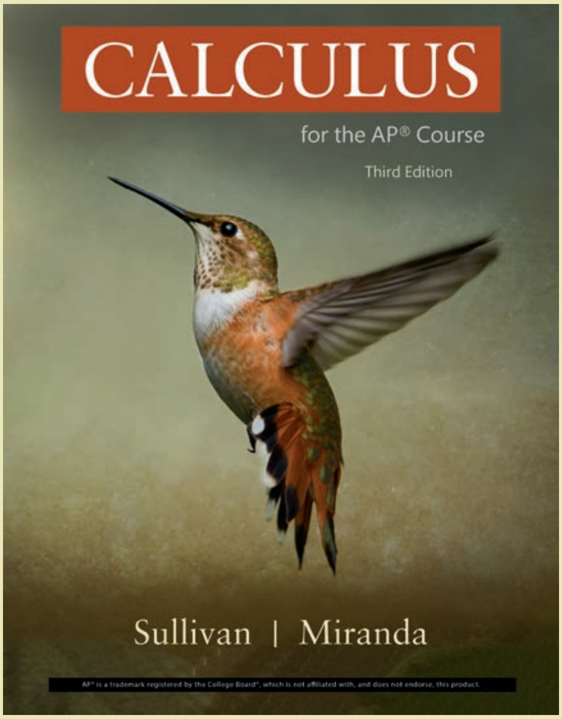

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